ние принадлежности решения к некоторому функциональному шару, центром которого является предельное с весом решение.

Ключевые слова: нелинейное дифференциальное уравнение, сингулярная краевая задача, предельное с весом решение, аппроксимация.

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NUMERICAL SOLUTION OF THE BOUNDARY VALUE PROBLEMS FOR THE LOADED DIFFERENTIAL AND FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS

Abstract. *The article presents a computational method to solve boundary value problems for the loaded differential and Fredholm integro-differential equations. Solving a problem for the loaded differential and Fredholm integro-differential equations is reduced to solving a system of linear algebraic equations in relation to the additional parameters introduced. A numerical method for finding a solution of the problem is suggested, which is based on solving the constructed system and the Bulirsch-Stoer method for solving Cauchy problems on the subintervals. The result is illustrated by example.*

Key words: *integro-differential equation, loaded differential equation, parametrization method, numerical method.*

Introduction. Loaded differential equations are used to solve problems of long-term prediction and control of the groundwater level and soil moisture [1, 2]. Various problems for loaded differential equations and methods of finding their solutions are considered in [1, 3-8].

A new concept of a general solution of a linear loaded differential equation was proposed in [9]. A new general solution was introduced for the linear Fredholm integro-differential equation in [10]. Replacing the integral term of an integro-differential equation with a quadrature formula also leads to a loaded differential equation. Therefore, numerical and approximate methods for solving boundary value problems for loaded differential equations are also used in solving boundary value problems for integro-differential equations.

On the basis of the parametrization method [11], in [10], a new approach to the general solution of the linear Fredholm integro-differential equation was proposed. The interval where the equation is considered is divided into parts, and the values of the solution at the starting points of subintervals are taken as additional parameters. With the help of newly introduced unknown functions on the sub-intervals, a special Cauchy problem for a system of integro-differential equations with parameters is received. Using the solution of the special Cauchy problem, a new general solution of the Fredholm linear integro-differential equation is constructed. This general solution, in contrast to the classical general solution, exists for all Fredholm linear integro-differential equations. The new general solution allowed us to propose numerical and approximate methods for solving linear boundary value problems for Fredholm integro-differential equations. The basis of these methods is to construct and solve systems of linear algebraic equations with respect to arbitrary vectors of the new general solution. Coefficients and right-hand sides of the systems are determined by solving the Cauchy problems for linear ordinary differential equations on sub-intervals and solving the linear Fredholm integral equation of the second kind. In [12-14], the linear ordinary Fredholm integral-differential equation is approximated by a loaded differential equation. The interrelation between the well-posed solvability of linear boundary value problems for the initial Fredholm integro-differential equation and the approximated loaded differential equation is established. The necessary and sufficient conditions for the well-posed solvability of a linear two-point boundary value problem for the system of Fredholm integro-differential equations, containing the derivative in the integral member, in terms of the fundamental matrix and the solvability of the second-kind Fredholm equation, were established in [15].

Despite the large number of papers devoted to the study and solving of boundary value problems for loaded differential and integro-differential equations, many questions related to the solvability of boundary value problems for the loaded differential and Fredholm integro-differential equations remain unsolved.

Statement of problem. Consider the boundary value problems for the loaded differential and Fredholm integro-differential equations

$$\frac{dx}{dt} = A_0(t)x + \sum_{k=1}^{m}\int_0^T \varphi_k(t)\psi_k(s)x(s)ds + \sum_{i=1}^{N} A_i(t)x(\theta_i) + f(t),\ t \in (0,T), \tag{1}$$

$$Bx(0) + Cx(T) = d, \quad d \in R^n, \quad x \in R^n, \tag{2}$$

where the $(n \times n)$-matrices $A_j(t)$, $j = \overline{0,N}$, $\varphi_k(t)$ and $\psi_k(\tau)$ are continuous on $[0,T]$, $k = \overline{1,m}$; the n-vector $f(t)$ is continuous on $[0,T]$; B and C are constant $(n \times n)$ - matrices, and $0 = \theta_0 < \theta_1 < \cdots < \theta_N < \theta_{N+1} = T$, $\|x\| = \max_{i=\overline{1,n}}|x_i|$.

Let $C([0,T],R^n)$ denote the space of continuous on $[0,T]$ functions $x(t)$ with norm $\|x\|_1 = \max_{t\in[0,T]}\|x(t)\|$.

The solution to problems (1) and (2) is a continuously differentiable on $(0,T)$ function $x(t) \in C([0,T],R^n)$ satisfying the system of loaded differential and Fredholm integro-differential equations (1) and boundary condition (2).

Scheme of parametrization method. Given the points: $0 = \theta_0 < \theta_1 < \cdots < \theta_N < \theta_{N+1} = T$, and let Δ_N denote the partition of interval $[0,T)$ into $N+1$ subintervals $[0,T) = \cup_{r=1}^{N+1}[\theta_{r-1},\theta_r)$.

Define the space $C([0,T],\Delta_N,R^{n(N+1)})$ of systems of functions $x[t] = (x_1(t),x_2(t),\dots,x_{N+1}(t))$, where $x_r:[\theta_{r-1},\theta_r) \to R^n$ are continuous on $[\theta_{r-1},\theta_r)$ and have finite left-sided limits $\lim_{t\to\theta_r-0} x_r(t)$ for all $r = \overline{1,N+1}$, with norm $\|x[\cdot]\|_2 = \max_{r=\overline{1,N+1}} \sup_{t\in[\theta_{r-1},\theta_r)}\|x_r(t)\|$.

The restriction of the function $x(t)$ to the r –th interval $[\theta_{r-1},\theta_r)$ is denoted by $x_r(t)$, i.e. $x_r(t) = x(t)$ for $t \in [\theta_{r-1},\theta_r)$, $r = \overline{1,N+1}$. Then we introduce additional parameters

$\lambda_r = x_r(\theta_{r-1}),\ r = \overline{1, N+1}$. Making the substitution $x_r(t) = u_r(t) + \lambda_r$ on every r -th interval $[\theta_{r-1}, \theta_r),\ \ r = \overline{1, N+1},$ we obtain the boundary value problem with parameters

$$\frac{du_r}{dt} = A_0(t)[u_r + \lambda_r] + \sum_{j=1}^{N+1}\sum_{k=1}^{m}\int_{\theta_{j-1}}^{\theta_j}\varphi_k(t)\psi_k(s)[u_j + \lambda_j]ds +$$

$$+\sum_{i=1}^{N} A_i(t)\lambda_{i+1} + f(t),\quad t \in [\theta_{r-1}, \theta_r),\quad r = \overline{1, N+1}, \tag{3}$$

$$u_r(\theta_{r-1}) = 0,\ r = \overline{1, N+1}, \tag{4}$$

$$B\lambda_1 + C\lambda_{N+1} + C\lim_{t\to T-0} u_{N+1}(t) = d, \tag{5}$$

$$\lambda_s + \lim_{t\to\theta_s-0} u_s(t) = \lambda_{s+1},\quad s = \overline{1, N}. \tag{6}$$

where (6) are conditions for matching the solution at the interior points of partition Δ_N.

A pair $(u^*[t], \lambda^*)$ with elements $u^*[t] = (u_1^*(t), u_2^*(t), \dots, u_{N+1}^*(t)) \in C([0,T], \Delta_N, R^{n(N+1)})$, $\lambda^* = (\lambda_1^*, \lambda_2^*, \dots, \lambda_{N+1}^*) \in R^{n(N+1)}$ is said to be a solution to problems (3)-(6) if the functions $u_r^*(t),\ r = \overline{1, N+1},$ are continuously differentiable on $[\theta_{r-1}, \theta_r)$ and satisfy (3) and additional conditions (5) and (6) with $\lambda_j = \lambda_j^*,\ j = \overline{1, N+1},$ and initial conditions (4).

Problems (1), (2) and (3)-(6) are equivalent. If the $x^*(t)$ is a solution of problems (1) and (2), then the pair $(u^*[t], \lambda^*)$, where $u^*[t] = (x^*(t) - x^*(\theta_0), x^*(t) - x^*(\theta_1), \dots, x^*(t) - x^*(\theta_N))$, and $\lambda^* = (x^*(\theta_0), x^*(\theta_1), \dots, x^*(\theta_N))$, is a solution of problems (3)-(6). Conversely, if a pair $(\tilde{u}[t], \tilde{\lambda})$ with elements $\tilde{u}[t] = (\tilde{u}_1(t), \tilde{u}_2(t), \dots, \tilde{u}_{N+1}(t)) \in C([0,T], \Delta_N, R^{n(N+1)})$, $\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \dots, \tilde{\lambda}_{N+1}) \in R^{n(N+1)}$, is a solution of (3)-(6), then the function $\tilde{x}(t)$ defined by the equalities $\tilde{x}(t) = \tilde{u}_r(t) + \tilde{\lambda}_r$, $t \in [\theta_{r-1}, \theta_r),\ \ r = \overline{1, N+1},$ and $\tilde{x}(T) = \lim_{t\to T-0}\tilde{u}_{N+1}(t) + \tilde{\lambda}_{N+1}$, will be the solution of the original problems (1) and (2).

Fixed λ_j problems (3) and (4) are special Cauchy problems for the system of Fredholm integro-differential equations. We have $N+1$ Cauchy problems on the intervals $[\theta_{r-1}, \theta_r)$, $r = \overline{1, N+1},$ and the system of integro-differential equations includes of the sum of integrals of all $N+1$ functions $u_r(t)$ with degenerate kernels on the segments $[\theta_{r-1}, \theta_r]$.

Using the fundamental matrix $X_r(t)$ of differential equation $\frac{dx}{dt} = A(t)x$ on $[\theta_{r-1}, \theta_r]$, we reduce the special Cauchy problem for the system of integro-differential equations with parameters (3) and (4) to the equivalent system of integral equations

$$u_r(t) = X_r(t)\int_{\theta_{r-1}}^{t} X_r^{-1}(\tau)A_0(\tau)\,d\tau\,\lambda_r + X_r(t)\int_{\theta_{r-1}}^{t} X_r^{-1}(\tau)\sum_{i=1}^{N} A_i(t)\lambda_{i+1}\,d\tau +$$

$$+X_r(t)\int_{\theta_{r-1}}^{t} X_r^{-1}(\tau)\sum_{j=1}^{N+1}\sum_{k=1}^{m}\int_{\theta_{j-1}}^{\theta_j}\varphi_k(\tau)\psi_k(s)[u_j(s) + \lambda_j]\,ds\,d\tau +$$

$$+X_r(t)\int_{\theta_{r-1}}^{t} X_r^{-1}(\tau) f(\tau)\,d\tau,\quad t \in [\theta_{r-1}, \theta_r),\ \ r = \overline{1, N+1}. \tag{7}$$

Let $\mu_k = \sum_{j=1}^{N+1}\int_{\theta_{j-1}}^{\theta_j}\psi_k(s)u_j(s)ds,\ \ k = \overline{1, m},$ and rewrite the system of integral equations (7) in the following form

$$u_r(t) = \sum_{k=1}^{m} X_r(t)\int_{\theta_{r-1}}^{t} X_r^{-1}(\tau)\varphi_k(\tau)\,d\tau\,\mu_k +$$

$$+X_r(t)\int_{\theta_{r-1}}^{t} X_r^{-1}(\tau)A_0(\tau)d\tau\,\lambda_r + X_r(t)\int_{\theta_{r-1}}^{t} X_r^{-1}(\tau)\sum_{i=1}^{N} A_i(t)\lambda_{i+1}d\tau +$$

$$+X_r(t)\int_{\theta_{r-1}}^{t} X_r^{-1}(\tau)\sum_{k=1}^{m}\varphi_k(\tau)\sum_{j=1}^{N+1}\int_{\theta_{j-1}}^{\theta_j}\psi_k(s)\lambda_j ds\,d\tau +$$

$$+X_r(t)\int_{\theta_{r-1}}^{t} X_r^{-1}(\tau)f(\tau)d\tau,\quad t\in[\theta_{r-1},\theta_r),\ r=\overline{1,N+1}. \tag{8}$$

Multiplying both sides of (8) by $\psi_p(t)$, integrating on the interval $[\theta_{r-1},\theta_r]$ and summing up over r, we get the system of linear algebraic equations with respect to $\mu=(\mu_1,\mu_2,\dots,\mu_m)\in R^{nm}$

$$\mu_p=\sum_{k=1}^{m}G_{p,k}(\Delta_N)\mu_k+\sum_{r=1}^{N+1}V_{p,r}(\Delta_N)\lambda_r+\sum_{j=1}^{N}W_{p,j}(\Delta_N)\lambda_{j+1}+g_p(f,\Delta_N),\ \ p=\overline{1,m}, \tag{9}$$

with the $(n\times n)$ matrices

$$G_{p,k}(\Delta_N)=\sum_{r=1}^{N+1}\int_{\theta_{r-1}}^{\theta_r}\psi_p(\tau)X_r(\tau)\int_{\theta_{r-1}}^{\tau}X_r^{-1}(s)\varphi_k(s)dsd\tau,\quad k=\overline{1,m},$$

$$V_{p,r}(\Delta_N)=\int_{\theta_{r-1}}^{\theta_r}\psi_p(\tau)X_r(\tau)\int_{\theta_{r-1}}^{\tau}X_r^{-1}(s)A_0(s)\,dsd\tau+\sum_{j=1}^{N+1}\sum_{k=1}^{m}\int_{\theta_{j-1}}^{\theta_j}\psi_p(\tau)\times$$

$$\times X_j(\tau)\int_{\theta_{j-1}}^{\tau}X_j^{-1}(\tau_1)\varphi_k(\tau_1)d\tau_1 d\tau\int_{\theta_{r-1}}^{\theta_r}\psi_k(s)ds,\quad r=\overline{1,N+1},$$

$$W_{p,j}(\Delta_N)=\sum_{r=1}^{N+1}\int_{\theta_{r-1}}^{\theta_r}\psi_p(\tau)X_r(\tau)\int_{\theta_{r-1}}^{\tau}X_r^{-1}(s)A_j(s)dsd\tau,\quad j=\overline{1,N},$$

and vectors of dimension n

$$g_p(f,\Delta_N)=\sum_{r=1}^{N+1}\int_{\theta_{r-1}}^{\theta_r}\psi_p(\tau)X_r(\tau)\int_{\theta_{r-1}}^{\tau}X_r^{-1}(s)f(s)dsd\tau,\ \ p=\overline{1,m},$$

Using the matrices $G_{p,k}(\Delta_N)$, $V_{p,r}(\Delta_N)$, $W_{p,j}(\Delta_N)$, form the $G(\Delta_N)=\left(G_{p,k}(\Delta_N)\right), p,k=\overline{1,m},$ and $V(\Delta_N)=\left(V_{p,r}(\Delta_N)\right),\ p=\overline{1,m},\ r=\overline{1,N+1}$, and $W(\Delta_N)=\left(W_{p,j}(\Delta_N)\right),\ p=\overline{1,m},\ j=\overline{1,N}$. Then the system (9) has the form

$$[I-G(\Delta_N)]\mu=V(\Delta_N)\lambda+W(\Delta_N)\xi+g(f,\Delta_N), \tag{10}$$

where I is the identity matrix of dimension nm, $\xi=(\lambda_2,\lambda_3,\dots,\lambda_{N+1})\in R^{nN}$ and

$$g(f,\Delta_N)=\left(g_1(f,\Delta_N),g_2(f,\Delta_N),\dots,g_m(f,\Delta_N)\right)\in R^{nm}.$$

Definition 1. Partition Δ_N is called regular if the matrix $I-G(\Delta_N)$ is invertible.

Let $\sigma(m,[0,T])$ denote the set of regular partitions Δ_N of $[0,T]$ for the equation (1).

Definition 2. The special Cauchy problems (3) and (4) is called uniquely solvable, if for any $\lambda\in R^{n(N+1)}$, $f(t)\in C([0,T],R^n)$ they have a unique solution.

Special Cauchy problems (3) and (4) is equivalent to the system of integral equations (7). This system by virtue of the kernel degeneracy is equivalent to the system of algebraic equations (9) with respect to $\mu=(\mu_1,\mu_2,\dots,\mu_m)\in R^{nm}$. Therefore, the special Cauchy problem is uniquely solvable if and only if the partition Δ_N, generating this problem, is regular.

Since the special Cauchy problem is uniquely solvable for the sufficiently small partition step $h>0$ (see [14]), the set $\sigma(m,[0,T])$ is not empty.

Take $\Delta_N\in\sigma(m,[0,T])$ and present $[I-G(\Delta_N)]^{-1}$ in the next form

$$[I-G(\Delta_N)]^{-1}=\left(M_{k,p}(\Delta_N)\right),\ k,p=\overline{1,m},$$

where $M_{k,p}(\Delta_N)$ are the square matrices of dimension n.

Then according to (10) the elements of vector $\mu \in R^{nm}$ can be determined by the equalities

$$\mu_k = \sum_{j=1}^{N+1}\Big(\sum_{p=1}^{m} M_{k,p}(\Delta_N)V_{p,j}(\Delta_N)\Big)\lambda_j + \sum_{j=1}^{N}\Big(\sum_{p=1}^{m} M_{k,p}(\Delta_N)W_{p,j}(\Delta_N)\Big)\lambda_{j+1} +$$
$$+\sum_{p=1}^{m} M_{k,p}(\Delta_N)g_p(f,\Delta_N)\,,\ k=\overline{1,m}, \qquad (11)$$

In (8), substituting the right-hand side of (11) instead of μ_k, we get the representation of functions $u_r(t)$ via λ_j, $j=\overline{1,N+1}$:

$$u_r(t) = \sum_{j=1}^{N+1}\left\{\sum_{k=1}^{m} X_r(t)\int_{\theta_{r-1}}^{t} X_r^{-1}(\tau)\varphi_k(\tau)d\tau \sum_{p=1}^{m} M_{k,p}(\Delta_N)V_{p,j}(\Delta_N)\right\}\lambda_j +$$
$$+\sum_{j=1}^{N+1}\left\{\sum_{k=1}^{m} X_r(t)\int_{\theta_{r-1}}^{t} X_r^{-1}(\tau)\varphi_k(\tau)d\tau \int_{\theta_{j-1}}^{\theta_j}\psi_k(s)ds\right\}\lambda_j +$$
$$+\sum_{j=1}^{N}\left\{\sum_{k=1}^{m} X_r(t)\int_{\theta_{r-1}}^{t} X_r^{-1}(\tau)\varphi_k(\tau)d\tau \sum_{p=1}^{m} M_{k,p}(\Delta_N)W_{p,j}(\Delta_N)\right\}\lambda_{j+1} +$$
$$+X_r(t)\int_{\theta_{r-1}}^{t} X_r^{-1}(\tau)A_0(\tau)d\tau\,\lambda_r + X_r(t)\int_{\theta_{r-1}}^{t} X_r^{-1}(\tau)\sum_{i=1}^{N} A_i(t)\lambda_{i+1}d\tau +$$
$$+X_r(t)\int_{\theta_{r-1}}^{t} X_r^{-1}(\tau)\left[\sum_{k=1}^{m}\varphi_k(\tau)\sum_{p=1}^{m} M_{k,p}(\Delta_N)g_p(f,\Delta_N) + f(\tau)\right]d\tau,$$
$$t\in[\theta_{r-1},\theta_r),\ \ r=\overline{1,N+1}. \qquad (12)$$

Introduce the notations:

$$D_{r,j}(\Delta_N) = \sum_{k=1}^{m} X_r(\theta_r)\int_{\theta_{r-1}}^{\theta_r} X_r^{-1}(\tau)\varphi_k(\tau)d\tau\Big[\sum_{p=1}^{m} M_{k,p}(\Delta_N)V_{p,j}(\Delta_N) +$$
$$+\int_{\theta_{j-1}}^{\theta_j}\psi_k(s)ds\Big],\ \ j\neq r,\ r,j=\overline{1,N+1},$$
$$D_{r,r}(\Delta_N) = \sum_{k=1}^{m} X_r(\theta_r)\int_{\theta_{r-1}}^{\theta_r} X_r^{-1}(\tau)\varphi_k(\tau)d\tau\Big[\sum_{p=1}^{m} M_{k,p}(\Delta_N)V_{p,r}(\Delta_N) +$$
$$+\int_{\theta_{r-1}}^{\theta_r}\psi_k(s)ds\Big] + X_r(\theta_r)\int_{\theta_{r-1}}^{\theta_r} X_r^{-1}(\tau)A_0(\tau)d\tau,\ \ r=\overline{1,N+1},$$
$$E_{r,j}(\Delta_N) = \sum_{k=1}^{m} X_r(\theta_r)\int_{\theta_{r-1}}^{\theta_r} X_r^{-1}(\tau)\varphi_k(\tau)d\tau\sum_{p=1}^{m} M_{k,p}(\Delta_N)W_{p,j}(\Delta_N) +$$
$$+X_r(\theta_r)\int_{\theta_{r-1}}^{\theta_r} X_r^{-1}(\tau)A_j(t)d\tau,\ \ j=\overline{1,N},$$
$$F_r(\Delta_N) = \sum_{k=1}^{m} X_r(\theta_r)\int_{\theta_{r-1}}^{\theta_r} X_r^{-1}(\tau)\Big[\varphi_k(\tau)\sum_{p=1}^{m} M_{k,p}(\Delta_N)g_p(f,\Delta_N) + f(\tau)\Big]d\tau,$$

Then from (12) we get

$$\lim_{t\to\theta_r-0} u_r(t) = \sum_{j=1}^{N+1} D_{r,j}(\Delta_N)\lambda_j + \sum_{j=1}^{N} E_{r,j}(\Delta_N)\lambda_{j+1} + F_r(\Delta_N). \qquad (13)$$

Substituting the right-hand side of (13) into the boundary condition (5) and conditions of matching solution (6), we have the following system of linear algebraic equations with respect to parameters λ_r, $r=\overline{1,N+1}$:

$$[B + CD_{N+1,1}(\Delta_N)]\lambda_1 + \sum_{j=2}^{N} CD_{N+1,j}(\Delta_N)\lambda_j + C[I + D_{N+1,N+1}(\Delta_N)]\lambda_{N+1} +$$
$$+\sum_{j=1}^{N} CE_{N+1,j}(\Delta_N)\lambda_{j+1} = d - CF_{N+1}(\Delta_N), \qquad (14)$$

$$[I + D_{p,p}(\Delta_N)]\lambda_p - [I - D_{p,p+1}(\Delta_N)]\lambda_{p+1} + \sum_{j=1}^{N} E_{p,j}(\Delta_N)\lambda_{j+1} = -F_p(\Delta_N),\ \ p=\overline{1,N}. \qquad (15)$$

Denoting by $Q_*(\Delta_N)$ the matrix corresponding to the left-hand side of the system of equations (14) and (15), we get

$$Q_*(\Delta_N)\lambda = -F_*(\Delta_N), \quad \lambda \in R^{n(N+1)}, \tag{16}$$

where $F_*(\Delta_N) = \left(-d + CF_{N+1}(\Delta_N), F_1(\Delta_N), \dots, F_N(\Delta_N)\right)$.

It is not difficult to establish that the solvability of the boundary value problems (6) and (7) is equivalent to the solvability of the system (16). The solution of the system (16) is a vector $\lambda^* = (\lambda_1^*, \lambda_2^*, \dots, \lambda_{N+1}^*) \in R^{n(N+1)}$ consists of the values of the solutions of the original problems (6) and (7) in the initial points of subintervals, i.e. $\lambda_r^* = x^*(\theta_{r-1})$, $r = \overline{1, N+1}$.

Further we consider the Cauchy problems for ordinary differential equations on subintervals

$$\frac{dz}{dt} = A(t)z + P(t), \quad z(\theta_{r-1}) = 0, \quad t \in [\theta_{r-1}, \theta_r], \quad r = \overline{1, N+1}, \tag{17}$$

where $P(t)$ is either $(n \times n)$ matrix, or n vector, both continuous on $[\theta_{r-1}, \theta_r]$, $r = \overline{1, N+1}$. Consequently, the solution to problem (17) is a square matrix or a vector of dimension n. Denote by $a(P, t)$ the solution to the Cauchy problem (17). Obviously,

$$a(P, t) = X_r(t) \int_{\theta_{r-1}}^{t} X_r^{-1}(\tau) \mathrm{P}(\tau)\, d\tau, \quad t \in [\theta_{r-1}, \theta_r],$$

where $X_r(t)$ is a fundamental matrix of differential equation (17) on the r-th interval.

Numerical implementation of the parametrization method. We offer the following numerical implementation of the parametrization method. The algorithm is based on the Bulirsch-Stoer method to solve the Cauchy problems for ordinary differential equations and it is based on Simpson's method for estimation of the definite integrals.

1. Suppose we have a partition: $0 = \theta_0 < \theta_1 < \dots < \theta_N < \theta_{N+1} = T$. Divide each r-th interval $[\theta_{r-1}, \theta_r]$, $r = \overline{1, N+1}$, into N_r parts with step $h_r = (\theta_r - \theta_{r-1})/N_r$. Assume on each interval $[\theta_{r-1}, \theta_r]$ the variable $\hat{\theta}$ takes its discrete values: $\hat{\theta} = \theta_{r-1}$, $\hat{\theta} = \theta_{r-1} + h_r, \dots$, $\hat{\theta} = \theta_{r-1} + (N_r - 1)h_r$, $\hat{\theta} = \theta_r$, and denote by $\{\theta_{r-1}, \theta_r\}$ the set of such points.

2. Using the Bulirsch-Stoer method, we find the numerical solutions to Cauchy problems (17) and define the values of $(n \times n)$ matrices $a_r^{h_r}(\varphi_k, \hat{\theta})$ on the set $\{\theta_{r-1}, \theta_r\}$, $r = \overline{1, N+1}$, $k = \overline{1, m}$.

3. Using the values of $(n \times n)$ matrices $\psi_k(s)$ and $a_r^{h_r}(\varphi_k, \hat{\theta})$ on $\{\theta_{r-1}, \theta_r\}$ and Simpson's method, we calculate the $(n \times n)$ matrices

$$\hat{\psi}_{p,r}^{h_r}(\varphi_k) = \int_{\theta_{r-1}}^{\theta_r} \psi_p(\tau) a_r^{h_r}(\varphi_k, \tau) d\tau, \quad p,k = \overline{1, m}, \quad r = \overline{1, N+1}.$$

Summing up the matrices $\hat{\psi}_{p,r}^{h_r}(\varphi_k)$ over r, we find the $(n \times n)$ matrices $G_{p,k}^{\tilde{h}}(\Delta_N) = \sum_{r=1}^{N+1} \hat{\psi}_{p,r}^{h_r}(\varphi_k)$, where $\tilde{h} = (h_1, h_2, \dots, h_{N+1}) \in R^n$. Using them, we compose the $nm \times nm$ matrix $G^{\tilde{h}}(\Delta_N) = \left(G_{p,k}^{\tilde{h}}(\Delta_N)\right)$, $p, k = \overline{1, m}$. Check the invertibility of matrix $[I - G^{\tilde{h}}(\Delta_N)]: R^{nm} \to R^{nm}$.

If this matrix is invertible, we find $[I - G^{\tilde{h}}(\Delta_N)]^{-1} = \left(M_{p,k}^{\tilde{h}}(\Delta_N)\right)$, $p, k = \overline{1, m}$. If it has no inverse, then we take a new partition. In particular, each subinterval can be divided into two.

4. Solving the Cauchy problems for ordinary differential equations

$$\frac{dz}{dt} = A_0(t)z + A_i(t), \quad z(\theta_{r-1}) = 0, \quad t \in [\theta_{r-1}, \theta_r], \quad i = \overline{0,N},$$

$$\frac{dz}{dt} = A_0(t)z + f(t), \quad z(\theta_{r-1}) = 0, \quad t \in [\theta_{r-1}, \theta_r], \quad r = \overline{1,N+1},$$

by using again the Bulirsch-Stoer method, we find the values of $(n \times n)$ matrices $a_r(A_0, \hat{\theta})$, $a_r(A_i, \hat{\theta})$, $i = \overline{1,N}$, and n vector $a_r(f, \hat{\theta})$ on $\{\theta_{r-1}, \theta_r\}$, $r = \overline{1,N+1}$.

5. Applying Simpson's method on the set $\{\theta_{r-1}, \theta_r\}$, we evaluate the definite integrals

$$\hat{\psi}_{p,r}^{h_r} = \int_{\theta_{r-1}}^{\theta_r} \psi_p(s)ds, \quad \hat{\psi}_{p,r}^{h_r}(A_i) = \int_{\theta_{r-1}}^{\theta_r} \psi_p(\tau) a_r^{h_r}(A_i,\tau)\,d\tau, \quad i = \overline{0,N},$$

$$\hat{\psi}_{p,r}^{h_r}(f) = \int_{\theta_{r-1}}^{\theta_r} \psi_p(\tau) a_r^{h_r}(f,\tau)\,d\tau, \quad p = \overline{1,m}, \quad r = \overline{1,N+1}.$$

By the equalities

$$V_{p,r}^{\tilde{h}}(\Delta_N) = \hat{\psi}_{p,r}^{h_r}(A_0) + \sum_{j=1}^{N+1}\sum_{k=1}^{m} \hat{\psi}_{p,j}^{h_j}(\varphi_k) \cdot \hat{\psi}_{p,r}^{h_r}, \quad r = \overline{1,N+1},$$

$$W_p^{\tilde{h}}(A_i, \Delta_N) = \sum_{r=1}^{N+1} \hat{\psi}_{p,r}^{h_r}(A_i), \; i = \overline{1,N}, \quad g_p^{\tilde{h}}(f, \Delta_N) = \sum_{r=1}^{N+1} \hat{\psi}_{p,r}^{h_r}(f), \quad p = \overline{1,m},$$

we define the $(n \times n)$ matrices $V_{p,r}^{\tilde{h}}(\Delta_N)$, $r = \overline{1,N+1}$, $W_p^{\tilde{h}}(A_i, \Delta_N)$, $i = \overline{1,N}$, and n vectors $g_p^{\tilde{h}}(f, \Delta_N)$, respectively, $p = \overline{1,m}$.

6. Construct the system of linear algebraic equations with respect to parameters

$$Q_*^{\tilde{h}}(\Delta_N)\lambda = -F_*^{\tilde{h}}(\Delta_N), \quad \lambda \in R^{n(N+1)}, \tag{18}$$

The elements of matrix $Q_*^{\tilde{h}}(\Delta_N)$ and vector $F_*^{\tilde{h}}(\Delta_N) = \left(-d + CF_{N+1}^{\tilde{h}}(\Delta_N), F_1^{\tilde{h}}(\Delta_N), \dots, F_N^{\tilde{h}}(\Delta_N)\right)$ are defined by the equalities $D_{r,j}(\Delta_N) = \sum_{k=1}^{m} a_r^{h_r}(\varphi_k, \theta_r)\left[\sum_{p=1}^{m} M_{k,p}^{\tilde{h}}(\Delta_N) V_{p,j}^{\tilde{h}}(\Delta_N) + \hat{\psi}_{p,j}^{h_j}\right]$, $j \neq r$, $r, j = \overline{1,N+1}$,

$$D_{r,r}^{\tilde{h}}(\Delta_N) = \sum_{k=1}^{m} a_r^{h_r}(\varphi_k, \theta_r)\left[\sum_{p=1}^{m} M_{k,p}^{\tilde{h}}(\Delta_N) V_{p,r}^{\tilde{h}}(\Delta_N) + \hat{\psi}_{p,r}^{h_r}\right] + a_r^{h_r}(A_0, \theta_r),$$

$$E_{r,j}^{\tilde{h}}(\Delta_N) = \sum_{k=1}^{m} a_r^{h_r}(\varphi_k, \theta_r) \sum_{p=1}^{m} M_{k,p}^{\tilde{h}}(\Delta_N) W_{p,j}^{\tilde{h}}(\Delta_N) + a_r^{h_r}(A_j, \theta_r), \; j = \overline{1,N},$$

$$F_r^{\tilde{h}}(\Delta_N) = \sum_{k=1}^{m} a_r^{h_r}(\varphi_k, \theta_r) \sum_{p=1}^{m} M_{k,p}^{\tilde{h}}(\Delta_N) g_p^{\tilde{h}}(\Delta_N) + a_r^{h_r}(f, \theta_r), \; r = \overline{1,N+1}.$$

Solving the system (18), we find $\lambda^{\tilde{h}}$. As noted above, the elements of $\lambda^{\tilde{h}} = (\lambda_1^{\tilde{h}}, \lambda_2^{\tilde{h}}, \dots, \lambda_{N+1}^{\tilde{h}})$ are the values of approximate solution to problems (1) and (2) in the starting points of subintervals: $x^{\tilde{h}_r}(\theta_{r-1}) = \lambda_r^{\tilde{h}}$, $r = \overline{1,N+1}$.

7. To define the values of approximate solution at the remaining points of set $\{\theta_{r-1}, \theta_r\}$, we first find

$$\mu_k^{\tilde{h}} = \sum_{j=1}^{N+1}\left(\sum_{p=1}^{m} M_{k,p}^{\tilde{h}}(\Delta_N) V_{p,j}^{\tilde{h}}(\Delta_N)\right)\lambda_j^{\tilde{h}} + \sum_{j=1}^{N}\left(\sum_{p=1}^{m} M_{k,p}^{\tilde{h}}(\Delta_N) W_{p,j}^{\tilde{h}}(\Delta_N)\right)\lambda_{j+1}^{\tilde{h}} +$$

$$+\sum_{p=1}^{m} M_{k,p}^{\tilde{h}}(\Delta_N) g_p^{\tilde{h}}(f, \Delta_N), \quad k = \overline{1,m},$$

and then solve the Cauchy problems

$$\frac{dx}{dt} = A(t)x + \mathcal{F}^{\tilde{h}}(t), \qquad x(\theta_{r-1}) = \lambda_r^{\tilde{h}}, \quad t \in [\theta_{r-1}, \theta_r], \quad r = \overline{1,N+1},$$

where $\mathcal{F}^{\tilde{k}}(t) = \sum_{k=1}^{m} \varphi_k(t) \left[\mu_k^{\tilde{k}} + \sum_{j=1}^{N+1} \hat{\psi}_{p,j}^{h_j} \lambda_j^{\tilde{k}}\right] + \sum_{i=1}^{N} A_i(t)\lambda_{i+1}^{\tilde{k}} + f(t).$

And the solutions to Cauchy problems are found by the Bulirsch-Stoer method. Thus, the algorithm allows us to find the numerical solution to the problems (1) and (2).

To illustrate the proposed approach for the numerical solving linear boundary value problems for the loaded differential and Fredholm integro-differential equations (1) and (2) on the basis of the parameterization method, let us consider the following example.

Example. We consider linear boundary value problems for the loaded differential and Fredholm integro-differential equations

$$\frac{dx}{dt} = A_0(t)x + \int_0^T \varphi_1(t)\psi_1(s)x(s)ds + A_1(t)x(\theta_1) + f(t), \; t \in (0,T), \quad (19)$$

$$Bx(0) + Cx(T) = d, \quad d \in R^2, \quad x \in R^2. \quad (20)$$

Here $\theta_0 = 0, \theta_1 = \frac{1}{2}, \theta_2 = T = 1, \quad A_0(t) = \begin{pmatrix} \sin t & 1 \\ t^2 & \cos t \end{pmatrix}, \; A_1(t) = \begin{pmatrix} 4 & t \\ e^t & 0 \end{pmatrix},$

$$\varphi_1(t) = \begin{pmatrix} t+1 & t^3 \\ 0 & t^2 \end{pmatrix}, \quad \psi_1(s) = \begin{pmatrix} s^2 & s \\ s-3 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 7 \\ 1 & -3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 3 & 7 \end{pmatrix},$$

$$d = \begin{pmatrix} -17 \\ -32 \end{pmatrix}, \quad f(t) = \begin{pmatrix} \frac{362t}{15} + 8\sin t - 6t^2 \sin t - t^3 \sin t - t^2 - \frac{247t^3}{60} + \frac{422}{15} \\ 8t + \frac{51e^t}{8} - 4t^2 \cos t + \frac{233t^2}{60} - 6t^4 - t^5 + 7t\cos t - 7 \end{pmatrix}.$$

We use the numerical implementation of the algorithm. The accuracy of the solution depends on the accuracy of solving the Cauchy problem on subintervals and evaluating definite integrals. We provide the results of the numerical implementation of the algorithm by partitioning the subintervals [0, 0.5] and [0.5, 1] with step h = 0.05.

The exact solution of problems (19) and (20) is $x^*(t) = \begin{pmatrix} t^3 + 6t^2 - 8 \\ 4t^2 - 7t \end{pmatrix}$.

Table 1 provides the $x^*(t_k)$, $k = \overline{0,20}$, exact solution values and $\tilde{x}(t_k), k = \overline{0,20}$, numerical solution values.

Table 1. Results received by using MathCad15

t	$\tilde{x}_1(t)$	$x_1^*(t)$	$\tilde{x}_2(t)$	$x_2^*(t)$
0	-8.00000313	-8	0.00000045	0
0.05	-7.98487798	-7.984875	-0.33999959	-0.34
0.1	-7.93900282	-7.939	-0.65999964	-0.66
0.15	-7.86162766	-7.861625	-0.95999969	-0.96
0.2	-7.75200248	-7.752	-1.23999974	-1.24
0.25	-7.6093773	-7.609375	-1.49999981	-1.5
0.3	-7.4330021	-7.433	-1.73999988	-1.74
0.35	-7.22212689	-7.222125	-1.95999995	-1.96
0.4	-6.97600167	-6.976	-2.16000004	-2.16
0.45	-6.69387643	-6.693875	-2.34000012	-2.34
0.5	-6.37500117	-6.375	-2.50000021	-2.5
0.55	-6.01862589	-6.018625	-2.64000031	-2.64
0.6	-5.62400059	-5.624	-2.7600004	-2.76
0.65	-5.19037526	-5.190375	-2.86000049	-2.86
0.7	-4.7169999	-4.717	-2.94000058	-2.94
0.75	-4.20312451	-4.203125	-3.00000066	-3
0.8	-3.64799908	-3.648	-3.04000072	-3.04
0.85	-3.05087361	-3.050875	-3.06000077	-3.06
0.9	-2.41099809	-2.411	-3.06000078	-3.06
0.95	-1.72762251	-1.727625	-3.04000077	-3.04
1	-0.99999687	-1	-3.0000007	-3

For the difference of the corresponding values of the exact and constructed solutions of the problem the following estimate is true:

$$\max_{j=\overline{0,20}} \|x^*(t_j) - \tilde{x}(t_j)\| < 0.000003.$$

Conclusion. In this work, we propose a numerical implementation of parametrization method for finding solutions to linear boundary value problems for the loaded differential and Fredholm integro-differential equations. Using the parametrization method, we reduce the considered problem to the equivalent boundary value problem with parameters. The unknown functions are determined from the Cauchy problems for the system of ordinary differential equations, and the introduced parameters are determined from the system of algebraic equations. A numerical algorithm for finding solution to the considered problem is constructed. The Cauchy problem is solved by the Bulirsch-Stoer method. The example illustrating the numerical algorithm of parametrization method is provided.

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Э.А. Бакирова, Ж.М. Кадирбаева

Жүктелген дифференциалдық және Фредгольм интегралдық-дифференциалдық теңдеулері үшін шеттік есептерді сандық шешу

Аңдатпа: Мақалада жүктелген дифференциалдық және Фредгольм интегралдық-дифференциалдық теңдеулері үшін шеттік есептерді шешудің есептеу әдісі ұсынылған. Жүктелген дифференциалдық және Фредгольм интегралдық-дифференциалдық теңдеулері үшін есептерді шешу енгізілген қосымша параметрлерге қатысты сызықтық алгебралық теңдеулер жүйесін шешуге келтіріледі. Ішкі интервалдарда Коши есебін шешу үшін Булирш-Штёр әдісі мен құрылған жүйені шешуге негізделген есептің шешімін табудың сандық әдісі берілген. Нәтиже мысалмен сипатталады.

Түйінді сөздер: интегралдық-дифференциалдық теңдеу, жүктелген дифференциалдық теңдеу, параметрлеу әдісі, сандық әдіс

Э.А. Бакирова, Ж.М. Кадирбаева

Численное решение краевых задач для нагруженных дифференциальных и интегро-дифференциальных уравнений Фредгольма

Аннотация. В статье представлен вычислительный метод решения краевых задач для нагруженных дифференциальных и интегро-дифференциальных уравнений Фредгольма. Решение задачи для нагруженных дифференциальных и интегро-дифференциальных уравнений Фредгольма сводится к решению системы линейных алгебраических уравнений относительно введенных дополнительных параметров. Предложен численный метод нахождения решения задачи, основанный на решении построенной системы и метода Булирша-Штёра для решения задачи Коши на подинтервалах. Результат иллюстрируется примером.

Ключевые слова: интегро-дифференциальное уравнение, нагруженное дифференциальное уравнение, метод параметризации, численный метод.

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