нашу жизнь в беспрецедентной степени. Очень трудно предотвратить эти события в краткосрочной перспективе, но план предотвращения рисков может уменьшить негативные последствия аварии. Настоящее исследование посвящено оценке потенциала наводнений в бассейне реки Малая Алматинка в Алматы с использованием четырех моделей прогнозирования случайного леса, линейной регрессии, дерево решений и градиентого бустинга.

Ключевые слова: прогнозирование наводнения, машинное обучение, метод случайного леса, метод линейной регрессии, метод дерева решений и метод градиентного бустинга.

About authors:

Symbat S. Kabdrakhova, PhD, 1) Senior lecturer of the Department of Computer Science of al-Farabi Kazakh National University 2) Leading researcher of the Department of Differential Equations, Institute of Mathematics and Mathematical Modeling.

Сведения об авторах:

Кабдрахова Сымбат Сейсенбековна, кандидат физико-математических наук; старший преподаватель кафедры информатики Казахского Национального университета имени аль-Фараби; ведущий научный сотрудник Института математики и математического моделирования, отдела дифференциальных уравнений.

Авторлар туралы мәліметтер:

Кабдрахова Сымбат Сейсенбековна, физика-математика ғылымдарының кандидаты, 1) Әл-Фараби атындағы Қазақ ұлттық университетінің информатика кафедрасының аға оқытушысы 2) Математика және математикалық модельдеу инстит

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Kokotova Ye.V.

K. Zhubanov Aktobe Regional State University, Aktobe, Kazakhstan

BOUNDED SOLUTIONS OF DIFFERENTIAL SYSTEMS WITH SINGULARITIES AND THEIR APPROXIMATIONS

Abstract. *Singular boundary value problems for a linear nonhomogeneous system of ordinary differential equations on a finite interval are considered. It is supposed that improper integrals of the norm of the coefficient matrix over semiaxes are infinite*

Key words: *ordinary differential equations, singular boundary value problem, bounded solution, approximation, behavior of solutions at singular points, the parameterization method.*

Numerous application problems give rise to differential equations on an infinite interval or with singularities at an endpoint. Various problems for such equations have been studied by many authors (see [1–8] and references therein). A survey of results on singular boundary value problems for second order ordinary differential equations, as well as examples of specific physical processes leading to them, can be found in [4].

It is known that one of the main issues of the theory of singular problems is the problem of their approximation by regular boundary value problems. The resolution of this problem allows us not only to construct an approximate method for finding solutions to singular boundary value problems, but also to establish effective criteria for their well-posedness in terms of approximating regular boundary value problems.

In [7,8], the questions of the existence of a unique solution of a linear differential equation bounded on the whole real line were studied by the parameterization method proposed by D. S. Dzhumabaev [9]. Approximating regular two-point boundary value problems were constructed to find the restriction of the bounded solution to a finite interval.

In the present paper, we consider the differential equation

$$\frac{dx}{dt} = A(t)x + f(t), \quad x \in R^n, \tag{1}$$

where $t \in (0,T)$, $A(t)$, $f(t)$ are continuous on $(0,T)$, $\tilde{\alpha}(t) = ||A(t)|| = \max_i \sum_{j=1}^n |a_{ij}(t)|$, $i = 1,2,\ldots,n$, is a function continuous on $(0,T)$ and satisfying the conditions $\lim_{a \to 0+0} \int_a^{T/2} \tilde{\alpha}(t)dt = \infty$, $\lim_{b \to T-0} \int_{T/2}^{b} \tilde{\alpha}(t)dt = \infty$.

Let $\tilde{C}(J,R^n)$ denote the space of functions $x: J \to R^n$ continuous and bounded on $J \subseteq (0,T)$ with the norm $||x||_1 = \sup_{t \in J} ||x(t)||$; by $\tilde{C}_{1/\alpha}(J,R^n)$ we denote the space of functions $f: J \to R^n$ continuous and bounded with the weight $1/\alpha(t)$, equipped with the norm $||f||_{1/\alpha} = \sup_{t \in J} \left\| \frac{f(t)}{\alpha(t)} \right\|$.

The problem of finding a solution of Eq. (1) bounded on $(0,T)$ when $f(t) \in \tilde{C}_{1/\alpha}((0,T),R^n)$ will be referred to as Problem 1_α.

In [10], Problem 1_α was studied using the parameterization method [9] with nonuniform partitioning of the interval $(0,T)$, where the partition points are chosen taking into account the values of the equation coefficients.

Let us take $\theta > 0$, $t_0 = \frac{T}{2}$, $\delta_0 > 0$ and make the partition $(0,T) = \bigcup_{r=-\infty}^{\infty} [t_{r-1}, t_r)$, where the points t_r, $r \in Z$, are determined from the relations $\int_{t_{r-1}}^{t_r} \alpha(t)dt = \theta$, $\alpha(t) = \max(\tilde{\alpha}(t), \delta_0)$.

Let $\bar{h}(\theta)$ denote a two-sided infinite sequence of numbers $h_r = t_r - t_{r-1}$, $r \in Z$, and m_n be the space of bounded two-sided infinite sequences $\lambda_r \in R^n$ with the norm

$$||\lambda||_2 = \|(\ldots, \lambda_r, \lambda_{r+1}, \ldots)'\|_2 = \sup_r ||\lambda_r||, \, r \in Z.$$

Definition. Problem 1_α is called well-posed if, for any $f(t) \in \tilde{C}_{1/\alpha}((0,T),R^n)$, it admits a unique solution $x(t) \in \tilde{C}((0,T),R^n)$, and the following inequality is valid: $\|x\|_1 \le K\|f\|_\alpha$, where K is a constant independent of $f(t)$. We call K the constant of well-posedness of Problem 1_α.

In [10], necessary and sufficient conditions for the well-posedness of Problem 1_α in terms of a two-sided infinite block band matrix $Q_{\nu,\bar{h}(\theta)}: m_n \to m_n$, of the form

$$Q_{\nu,\bar{h}(\theta)} = \left\| \begin{matrix} \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & 0 & I + D_{\nu,r}(h_r) & -I & 0 & 0 & \ldots \\ \ldots & 0 & 0 & I + D_{\nu,r+1}(h_{r+1}) & -I & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{matrix} \right\|,$$

where I is the identity matrix of order n,

$$D_{\nu,r}(h_r) = \sum_{j=0}^{\nu-1} \int_{t_{r-1}}^{t_r} A(\tau_1) \ldots \int_{t_{r-1}}^{\tau_j} A(\tau_{j+1}) d\tau_{j+1} \ldots d\tau_1, \quad \tau_0 = t_r, \; r \in Z.$$

In [11], the problem of finding an approximate solution to Problem 1_α is studied, to which we refer to as **Problem 2_α**. Given $\varepsilon > 0$, it is required to determine numbers $T_1, T_2 \in (0, T)$, real $(n \times n)-$matrices B, C, and $n-$vector d, such that the solution $x_{T_1,T_2}(t)$ to the two-point boundary value problem

$$\frac{dx}{dt} = A(t)x + f(t),\ t \in (T_1, T_2),\ x \in R^n, \tag{2}$$

$$Bx(T_1) + Cx(T_2) = d, \tag{3}$$

satisfies the inequality $\max_{x \in [T_1, T_2]} \left\| x_{T_1,T_2}(t) - x^*(t) \right\| < \varepsilon$, where $x^*(t)$ is a solution to Problem 1_α.

Problem 2_α is studied under the following assumptions:

1. The relations

$$\lim_{t \to 0+0} \frac{A(t)}{\alpha(t)} = A_0,\ \lim_{t \to T-0} \frac{A(t)}{\alpha(t)} = A_T;\ \operatorname{Re} \xi_i^0 \neq 0, \operatorname{Re} \xi_i^T \neq 0,$$

hold, where ξ_i^0 and ξ_i^T, $i = \overline{1,n}$ are the eigenvalues of the matrices A_0 and A_T, respectively.

2. $\lim_{t \to 0+0} \frac{f(t)}{\alpha(t)} = f_0,\ \lim_{t \to T-0} \frac{f(t)}{\alpha(t)} = f_T.$

The construction of approximating regular boundary value problems and establishing a mutual relationship between the well-posedness of the original problem and that of approximating problems in [11], as well as in [10], were carried out using the parameterization method with nonuniform partitioning.

In the present paper we study the behavior of the solution of Eq. (1) near the singular points. We consider the following problem.

Problem 3_α. For given functions $\beta_0(t)$ and $\beta_T(t)$ continuous on $(0; T/2]$ and $[T/2, T)$, respectively, it is required to find a solution $x^*(t)$ of Eq. (1) satisfying the following conditions:

$$\lim_{t \to 0+0} \| x^*(t) - \beta_0(t) \| = 0,\ \lim_{t \to T-0} \| x^*(t) - \beta_T(t) \| = 0. \tag{4}$$

In order to investigate Problem 3_α, we introduce the concept of a "limit solution" to Eq. (1) with weight $1/\alpha(t)$ as $t \to 0+0$ ($t \to T-0$).

Definition. A function $x_T(t)$ continuously differentiable on $[T/2, T)$ is called the limit solution to Eq. (1) with weight $1/\alpha(t)$ as $t \to T-0$, if

$$\lim_{t \to T-0} \left\| \frac{\dot{x}_T(t) - A(t)x_T(t) - f(t)}{\alpha(t)} \right\| = 0.$$

The limit solution $x_0(t)$ of Eq. (1) with weight $1/\alpha(t)$ as $t \to 0+0$ is defined in an analogous way.

Theorem 1. *Let Assumption **1** be fulfilled and* $\lim_{t \to T-0} \left\| \frac{f(t)}{\alpha(t)} \right\| = 0$. *Then all solutions* $x(t)$ *of Eq. (1) bounded on* $[T/2, T)$ *satisfy the equation* $\lim_{t \to T-0} \| x(t) \| = 0.$

Let $x_T(t)$ be a limit solution of Eq. (1) with weight $1/\alpha(t)$ as $t \to T-0$. We denote the set of solutions of Eq. (1) satisfying the condition $x(t) - x_T(t) \in \tilde{C}([T/2; T), R^n)$ by $X_T([T/2, T))$. The following theorem establishes an attracting property of the limit solution with weight $1/\alpha(t)$ as $t \to T-0$.

Theorem 2. *Let Problem* 1_α *be well-posed and* $x_T(t)$ *be a limit solution of Eq. (1) with weight* $1/\alpha(t)$ *as* $t \to T-0$. *Then* $X_T([T/2,T]) \neq \varnothing$, *and any solution* $x(t)$ *of Eq. (1) belonging to* $X_T([T/2,T])$ *satisfies the equation* $\lim_{t\to T-0} \| x(t) - x_T(t) \| = 0.$

Note that analogous theorems hold true for the limit solution as $t \to 0+0$.

Suppose that Problem 1_α is well-posed and there exist limit solutions $x_0(t)$ and $x_T(t)$ of Eq. (1) with weight $1/\alpha(t)$ as $t \to 0+0$ and $t \to T-0$, respectively, that satisfy the conditions

$$\lim_{t\to 0+0} \| x_0(t) - \beta_0(t) \| = 0, \quad \lim_{t\to T-0} \| x_T(t) - \beta_T(t) \| = 0 \cdot \qquad (5)$$

Let us choose some numbers $T_1 \in (0, T/2]$, $T_2 \in [T/2, T)$, a function $\bar{x}(t)$ continuously differentiable on $[T_1, T_2] \subset (0,T)$ that satisfies the conditions $\bar{x}(T_1) = x_0(T_1)$, $\bar{x}(T_2) = x_T(T_2)$, and construct the following function continuously differentiable on $(0,T)$:

$$\tilde{x}(t) = \begin{cases} x_0(t), & t \in (0, T_1], \\ \bar{x}(t), & t \in [T_1, T_2]. \\ x_T(t), & t \in [T_2, T) \end{cases}$$

By the substitution $y(t) = x(t) - \tilde{x}(t)$ we get

$$\frac{dy}{dt} = A(t)y + \tilde{F}(t), \; t \in (0,T), \; y \in R^n, \qquad (6)$$

where $\tilde{F}(t) = -\dot{\tilde{x}}(t) + A(t)\tilde{x}(t) + f(t) \in \tilde{C}_{1/\alpha}((0,T), R^n)$ and $\lim_{t\to 0+0} \left\| \frac{\tilde{F}(t)}{\alpha(t)} \right\| = 0, \; \lim_{t\to T-0} \left\| \frac{\tilde{F}(t)}{\alpha(t)} \right\| = 0.$

Since Problem 1_α is well-posed, Eq. (6) admits a unique solution $y^*(t)$ bounded on $(0,T)$. By Theorem 1 we get

$$\lim_{t\to 0+0} \left\| y^*(t) \right\| = 0, \lim_{t\to T-0} \| y^*(t) \| = 0. \qquad (7)$$

It follows from (5)-(7) that the function $x^*(t) = y^*(t) + \tilde{x}(t)$ satisfies Eq. (1) and relations (4).

Assume that $\hat{x}(t)$ is another solution of Problem 3_α. Then the function $x^*(t) - \hat{x}(t)$ is a bounded on $(0,T)$ solution of the homogeneous equation $\frac{dx}{dt} = A(t)x.$ Due to the well-posedness of Problem 1_α we have $x^*(t) = \hat{x}(t),$ meaning that- Problem 3_α has only one solution. Thus, the following statement holds true.

Theorem 3. *Let Problem* 1_α *be well-posed and there exist limit solutions* $x_0(t)$ *and* $x_T(t)$ *of Eq. (1) with weight* $1/\alpha(t)$ *as* $t \to 0+0$ *and* $t \to T-0$, *respectively, that satisfy conditions (5). Then Problem* 3_α *has a unique solution.*

Let us note in conclusion that under assumptions 1 and 2 and conditions of Theorem 3 for the coefficients and the right-hand side of Eq. (6), the regular two-point boundary value problem

$$\frac{dy}{dt} = A(t)y + \tilde{F}(t), \qquad t \in (T_1, T_2), \qquad (8)$$

$$P_1 S_0 A_0 y(T_1) + P_2 S_T A_T y(T_2) = d, \qquad (9)$$

approximating the problem of finding a solution of Eq. (6) bounded on $(0,T)$ (see Theorem 2 in [11]), allows one, with a given accuracy, to determine a restriction of the solution of problem 3_α to any interval $[T_1,T_2]\subset(0,T)$. Here S_0 and S_T are real nonsingular $(n\times n)$ matrices that reduce the matrices A_0 and A_T , respectively, to the generalized Jordan form $\tilde{A}_0 = S_0 A_0 S_0^{-1} = \begin{Vmatrix} A_{11}^0 & 0 \\ 0 & A_{22}^0 \end{Vmatrix}$, $\tilde{A}_T = S_T A_T S_T^{-1} = \begin{Vmatrix} A_{11}^T & 0 \\ 0 & A_{22}^T \end{Vmatrix}$.

Here A_{11}^0 and A_{22}^0 (A_{11}^T and A_{22}^T) consist of generalized Jordan boxes corresponding to the eigenvalues of the matrices A_0 (A_T) with negative and positive real parts, respectively; we denote their numbers by n_1^0 and n_2^0 (n_1^T and n_2^T); $P_1 = \begin{Vmatrix} I_{n_1} & 0 \\ 0 & 0 \end{Vmatrix}$, $P_2 = \begin{Vmatrix} 0 & 0 \\ 0 & I_{n_2} \end{Vmatrix}$, where I_{n_r}, $r=1,2$, are the identity matrices of order $n_r = n_r^0 = n_r^T$, $r=1,2$.

Taking into account $x^*(t) = y^*(t) + \tilde{x}(t)$, we can conclude that the approximation estimate for Problem 3_α depends on that for $y^*(t)$ by the solutions of two-point boundary value problems (8) and (9), where $\tilde{F}(t)$ and the right-hand side of (9) are determined via $\tilde{x}(t)$.

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Кокотова Е.В.
Сингулярлы дифференциалдық жүйелердің шектелген шешімдері және олардың аппроксимациялары

Аңдатпа: Шектелген аралықтағы жай дифференциалдық теңдеулердің сызықты біртекті емес жүйесі үшін сингулярлық шекаралық есептер қарастырылады. Жартылай аралықтарда коэффициенттер матрицасының нормасынан алынған меншіксіз интегралдар шексіз деп ұйғарылған.

Түйінді сөздер: жай дифференциалдық теңдеулер, сингулярлық шекаралық есеп, шектелген шешім, аппроксимациялар, ерекше нүктелердегі шешімдердің әрекеті, параметрлеу әдісі

Кокотова Е.В.
Ограниченные решения дифференциальных систем с сингулярностями и их аппроксимации

Аннотация. Рассматриваются сингулярные краевые задачи для линейной неоднородной системы обыкновенных дифференциальных уравнений на конечном интервале. Предполагается, что на полуинтервалах несобственные интегралы от нормы матрицы коэффициентов бесконечны.

Ключевые слова: обыкновенные дифференциальные уравнения, сингулярная краевая задача, ограниченное решение, аппроксимации, поведение решений в особых точках, метод параметризации.

About the author:

Yelena V. Kokotova-Candidate of Physical and Mathematical Sciences, Associate Professor, Department of MathematicsK. Zhubanov Aktobe Regional State University

Сведения об авторе:

Кокотова Елена Викторовна, кандидат физико-математических наук, доцент кафедры математики, Актюбинский региональный государственный университет им. К. Жубанова.

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S.T. Mynbayeva[1,2,*], S.G. Karakenova[3]

[1]Institute of Mathematics and Mathematical Modeling CS MES RK, Almaty,
[2]K. Zhubanov Aktobe Regional University, Aktobe,
[3]Al-Farabi Kazakh National University, Almaty

AN APPROACH TO SOLVING A NONLINEAR BOUNDARY VALUE PROBLEM FOR A FREDHOLM INTEGRO-DIFFERENTIAL EQUATION

Summary*. A nonlinear boundary value problem for a Fredholm integro-differential equation is considered. The interval where the problem is considered is partitioned and the values of a solution to the problem at the left endpoints of the subintervals are introduced as additional parameters. The introduction of additional parameters gives initial values at the left endpoints of subintervals for new unknown functions. The*