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ON A BOUNDARY VALUE PROBLEM FOR SYSTEMS OF DIFFERENTIAL EQUATIONS WITH SINGULARITIES

Abstract. *The parametrization method is used to investigate a linear two-point boundary value problem for a system of differential equations with singularities. The necessary conditions for the existence of a solution to the problem under consideration are established and an algorithm for finding their solution is proposed.*

Key words: *Boundary value problem, system of differential equations, conformable derivative, conformable integral, the parametrization method.*

Introduction.

In [1-10], definitions and basic properties of the conformable derivative were introduced.

Definition 1. Let the function $f:[0,\infty)\to R$. Then, for all $t>0$, the conformable derivative of the function f is defined as

$$T_\alpha(f)(t)=\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon},$$

where $\alpha\in(0,1)$. If f is differentiable in α order in $(0,a)$, $a>0$, and there is a $\lim\limits_{\varepsilon\to 0+} f^{(\alpha)}(t)$ then

$$f^{(\alpha)}(0)=\lim_{\varepsilon\to 0+} f^{(\alpha)}(t).$$

Definition 2. The Conformable integral of the f function of order $\alpha\in(0,1]$ is defined by the equality

$$I_\alpha^a(f)(t)=\int_a^t \tau^{\alpha-1}f(\tau)d\tau.$$

Lemma 1. Let for $t>0$ the functions f and g be differentiable in the order $\alpha\in(0,1)$. Then

1) $T_\alpha(af+bg)=aT_\alpha(f)+bT_\alpha(g)$, for all $a,b\in R$.
2) $T_\alpha(c)=0$, for all $f(t)=const$.
3) $T_\alpha(fg)=fT_\alpha(f)+T_\alpha(f)g$.
4) $T_\alpha\left(\frac{f}{g}\right)=\frac{fT_\alpha(f)-T_\alpha(f)g}{g^2}$.
5) If f is differentiable, then $T_\alpha(f)(t)=t^{1-\alpha}\frac{df}{dt}(t)$.

Lemma 2. Let $\alpha\in(0,1]$ and f functions be continuous under $t>a$, then

$$T_\alpha I_\alpha^a(f)(t)=f(t).$$

In this paper we consider a two point boundary value problem for systems of differential equations with a conformable derivative on the segment $[0,T]$

$$T_\alpha(x)(t)=A(t)x+f(t),\quad t\in[0,T], \tag{1}$$

$$Bx(0)+Cx(T)=d,\ d\in R^n, \tag{2}$$

where $(n\times n)$-matrix $t^{\alpha-1}A(t)$ and n-dimensional vector-function $f(t)$ are continuous on $[0,T]$,

$$\| A(t) \| = \max_{t\in[0,T]} \left\{ \max_i \sum_{j=1}^{n} |a_{ij}(t)| \right\} \le a \cdot$$

We need to find a vector function $[0,T]$ that is continuous on $(0,T)$ and continuously differentiable on $x(t)$ that satisfies the system of differential equations (1) and boundary conditions (2).

The boundary value problems (1) and (2) are investigated by using the parametrization method proposed by Professor D. Dzhumabaev [11]. Based on this method, the necessary conditions for the solvability of the problem under study are established and an algorithm for finding a solution is proposed.

Methods.

We take the step $h > 0$, which N times fits on the segment $[0,T]$ and we will produce splitting $[0,T) = \bigcup_{r=1}^{N} [(r-1)h, rh) \cdot$

The narrowing of the function $x(t)$ to $r-$the interval $[(r-1)h, rh)$ is denoted by $x_r(t)$, i.e., $x_r(t)$ is a system of vector functions defined and coinciding with $x(t)$ on $[(r-1)h, rh)$. Then the original two point boundary value problem for systems of differential equations will be reduced to an equivalent multi point boundary value problem

$$T_{\alpha.r}(x_r)(\mathrm{t}) = A(t)x_r + f(t), \quad t \in [(r-1)h, rh), \tag{3}$$

$$Bx_1(0) + C \lim_{t\to T-0} x_N(t) = d, \tag{4}$$

$$\lim_{t\to sh-0} x_s(t) = x_{s+1}(sh), \quad s = \overline{1, N-1}\ . \tag{5}$$

Here (5) are the gluing conditions at the inner points of the split $t = jh,$, $j = \overline{1, N-1}$.

If the function $x(t)$ is the solution of problems (1) and (2), the system contractions $x[t] = (x_1(t), x_2(t), ..., x_N(t))'$ will be the solution of multipoint boundary value problems (3) - (5). Conversely, if the system of vector functions $\tilde{x}[t] = (\tilde{x}_1(t), \tilde{x}_2(t), ..., \tilde{x}_N(t))'$ is the solution of problems (3) - (5), the function $\tilde{x}(t)$ defined by the equalities $\tilde{x}(t) = \tilde{x}_r(t)$, $t \in [(r-1)h, rh)$, $r = \overline{1, N}$, $\tilde{x}(T) = \lim_{t\to T-0} \tilde{x}_N(t)$ is the solution to the initial boundary value problems (1) and (2).

Using λ_r, we denote the value of the $x_r(t)$ functions at the $t = (r-1)h$ point and replace $[(r-1)h, rh)$ at each interval of $x_r(t) = u_r(t) + \lambda_r$, $r = \overline{1, N}$. then problems (3) - (5) are reduced to an equivalent multipoint boundary value problem with parameters

$$T_{\alpha,r}(u_r)(t) = A(t)[u_r + \lambda_r] + f(t), \tag{6}$$

$$u_r[(r-1)h] = 0,\ t \in [(r-1)h, rh),\ r = \overline{1, N}, \tag{7}$$

$$B\lambda_1 + C\lambda_N + C \lim_{t\to T-0} u_N(t) = d, \tag{8}$$

$$\lambda_s + \lim_{t\to sh-0} u_s(t) = \lambda_{s+1},\ s = \overline{1, N-1}. \tag{9}$$

Tasks (3) to (5) and (6) to (9) are equivalent in the sense that if the system functions $x[t] = (x_1(t), x_2(t), ..., x_N(t))'$ is the solution of problems (3) - (5), the pair $(\lambda, u[t])$ where

$\lambda=(x_1(0),x_2(h),...,x_N(N-1)h))'$, $u[t]=(x_1(t)-x_1(0),x_2(t)-x_2(h),...,x_N(t)-x_N((N-1)h))$, will be the solution of problem (6) - (9). Conversely, if the pair $(\tilde{\lambda},\tilde{u}[t])$ where $\tilde{\lambda}=(\tilde{\lambda}_1,\tilde{\lambda}_2,...,\tilde{\lambda}_N)'$, $\tilde{u}[t]=(\tilde{u}_1(t),\tilde{u}_2(t),...,\tilde{u}_N(t))'$ is the solution of problems (6)-(9), the system functions $\tilde{x}[t]=(\tilde{\lambda}_1+\tilde{u}_1(t),\tilde{\lambda}_2+\tilde{u}_2(t),...,\tilde{\lambda}_N+\tilde{u}_N(t))'$, are the solution of problems (3)-(5).

The appearance of the initial conditions $u_r[(r-1)h]=0,\ r=\overline{1,N},$ allow for fixed values of $\lambda=(\lambda_1,\lambda_2,...,\lambda_N)'$ to determine the functions of $u_r(t),\ r=\overline{1,N},$ from systems of integral equations

$$u_r(t)=\int_{(r-1)h}^{t}\tau^{\alpha-1}A(\tau)u_r(\tau)d\tau+\int_{(r-1)h}^{t}\tau^{\alpha-1}A(\tau)\lambda_r d\tau+\int_{(r-1)h}^{t}\tau^{\alpha-1}f(\tau)d\tau,\qquad t\in[(r-1)h,rh),\ r=\overline{1,N}. \tag{10}$$

From (10) defining $\lim\limits_{t\to Nh-0}u_N(t),\ \lim\limits_{t\to sh-0}u_s(t),\ s=\overline{1,N-1},$ substituting their corresponding expressions in conditions (8) and (9) and multiplying both parts (8) by $h>0$, we get a system of linear equations with respect to unknown parameters $\lambda_r,\ r=\overline{1,N}$:

$$hB\lambda_1+hC\lambda_N+hC\int_{(N-1)h}^{Nh}\tau^{\alpha-1}A(\tau)d\tau\lambda_N=hd-hC\int_{(N-1)h}^{Nh}\tau^{\alpha-1}f(\tau)d\tau-hC\int_{(N-1)h}^{Nh}\tau^{\alpha-1}A(\tau)u_N(\tau)d\tau, \tag{11}$$

$$\lambda_s+\int_{(s-1)h}^{sh}\tau^{\alpha-1}A(\tau)\lambda_s d\tau-\lambda_{s+1}=-\int_{(s-1)h}^{sh}\tau^{\alpha-1}A(\tau)u_N(\tau)d\tau-\int_{(s-1)h}^{sh}\tau^{\alpha-1}f(\tau)d\tau,\qquad s=\overline{1,N-1}. \tag{12}$$

The matrix of dimension $nN\times nN$ corresponding to the left side of the systems of linear equations (11) and (12) is denoted by $Q(h)$. Then the system of linear equations (11) and (12) is written as

$$Q(h)\lambda=-F(h)-G(u,h),\quad \lambda\in R^{nN}, \tag{13}$$

where

$$F(h)=\left(-hd+hC\int_{(N-1)h}^{Nh}\tau^{\alpha-1}f(\tau)d\tau,\int_0^h\tau^{\alpha-1}f(\tau)d\tau,\ \ldots,\int_{(N-2)h}^{(N-1)h}\tau^{\alpha-1}f(\tau)d\tau\right),$$

$$G(u,h)=\left(hC\int_{(N-1)h}^{Nh}\tau^{\alpha-1}A(\tau)u_N d\tau,\int_0^h\tau^{\alpha-1}A(\tau)u_1(\tau)d\tau,\ldots,\int_{(N-2)h}^{(N-1)h}\tau^{\alpha-1}A(\tau)u_{N-1}(\tau)d\tau\right)$$

Thus, to find the unknown pairs $(\lambda,u[t])$, the solution of problems (6)-(9), we have a closed system of equations (10) and (13). The solution of multipoint boundary value problems (6)-(9) is found as the limit of the sequence of pairs $(\lambda'^{(k)},u^{(k)}[t])$, $k=0,1,2,...$ determined by the following algorithm:

Step 0. a) Assuming that the $Q(h)$ matrix is invertible, we define the initial approximation from the $Q(h)\lambda=-F(h)$ equation using the $\lambda^{(0)}=(\lambda_1^{(0)},\lambda_2^{(0)},\ldots,\lambda_N^{(0)})\in R^{nN}$ parameter :

$$\lambda^{(0)}=-[Q(h)]^{-1}F(h).$$

b) Substituting the $\lambda_r^{(0)},\ r=\overline{1,N}$ found in the right part of the system of integro-differential equations (6) and solving the Cauchy problem with conditions (7) we find the $u^{(0)}[t]=(u_1^{(0)}(t),u_2^{(0)}(t),\ldots,u_N^{(0)}(t))'$.

Step 1. a) Substituting the found $u_r^{(0)}(t)$, $r=\overline{1,N}$, in the right part (13) of the equation $[Q(h)]\lambda=-F(h)-G(u^{(0)},h)$, we define $\lambda^{(1)}=(\lambda_1^{(1)},\lambda_2^{(1)},\ldots\lambda_N^{(1)})$.

b) Substituting the found $\lambda_r^{(1)}$, , $r=\overline{1,N}$ in the right part of the system of integro-differential equations (6) and solving the Cauchy problem with conditions (7) we find $u^{(1)}[t]=(u_1^{(1)}(t),u_2^{(1)}(t),\ldots,u_N^{(1)}(t))'$., etc.

Continuing the process, by following the algorithm we find a system of pairs $(\lambda^{(k)},u^{(k)}[t])$, $k=0,1,2,\ldots$

Based on the proposed algorithm for finding the solution, as well as theorem 1 of [11, page 53], it follows:

Theorem 1. *Let the* $Q(h)$ *matrix be invertible for some* $h>0: Nh=T$ *and the inequalities are satisfied:*

$$\|[Q(h)]^{-1}\|\leq\gamma(h),\quad q(h)=\gamma(h)\max(1,\frac{h^\alpha}{\alpha}\|C\|)\left[\exp\left(\frac{ah^\alpha}{\alpha}\right)-1\right]<1.$$

Then the two-point boundary value problems (1) and (2) have a unique solution.

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Ерекшелігі бар дифференциалдық теңдеулер жүйесі үшін шеттік есеп

Аңдатпа: Параметрлеу әдісімен ерекшеліктері бар дифференциалдық теңдеулер жүйесі үшін сызықтық екі нүктелі шеттік есеп зерттеледі. Қарастырылып отырған мәселені шешудің қажетті шарттары белгіленді және олардың шешімін табу алгоритмі ұсынылды.

Түйінді сөздер: Шеттік есеп, дифференциалдық теңдеулер жүйесі, конформды туынды, конформды интеграл, параметрлеу әдісі

К. Назарова, К. Усманов, Л. Асылханова

О краевой задаче для систем дифференциальных уравнений с особенностями

Аннотация. Методом параметризации исследуется линейная двухточечная краевая задача для системы дифференциальных уравнений с особенностями. Установлены необходи-

мые условия существования решения рассматриваемой задачи и предложен алгоритм нахождения их решения.

Ключевые слова: краевые задачи, система дифференциальных уравнений, конформная производная, конформный интеграл, метод параметризации.

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ON ONE PROPERTY OF A SOLUTION OF A THIRD ORDER PSEUDOPARABOLIC EQUATION

Abstract. *Sufficient conditions for the boundedness of the highest derivative of a third-order partial differential equation are established and a coercive estimate is obtained in the norm of the space* $C_*(\overline{\Omega}, R)$.

Key words: *pseudoparabolic equation, diagonal dominance, coercive estimate.*

1. **Introduction.** Let $\overline{\Omega} = [0,\omega]\times(-\infty,+\infty)$. Consider the third-order pseudoparabolic equation

$$u_{xtt} = a_0(x,t)u_{xt} + a_1(x,t)u_x + a_2(x,t)u_t + a_3(x,t)u + f(x,t) \quad (1)$$

where the functions $a_i(x,t)\,(i=\overline{0,3})$, $f(x,t)$ are assumed to be continuous and, generally speaking, unbounded on $\overline{\Omega}$.

At present, studies of local and nonlocal boundary value problems for equation (1) are very actively studied and arouse great practical and theoretical interest due to the fact that applied problems of physics, mechanics, and biology are reduced to such equations.

Local and nonlocal boundary value problems for pseudoparabolic equations of the third order are investigated in the works of M.Kh. Shkhanukov. [1-4]. A.M.Nakhushev [5-8] pointed out examples of the practical application of the results of the study of boundary value problems for the equations under study in the study of moisture transfer processes in porous media and in problems of mathematical biology.