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Кокотова Е.В.
Сингулярлы дифференциалдық жүйелердің шектелген шешімдері және олардың аппроксимациялары

Аңдатпа: Шектелген аралықтағы жай дифференциалдық теңдеулердің сызықты біртекті емес жүйесі үшін сингулярлық шекаралық есептер қарастырылады. Жартылай аралықтарда коэффициенттер матрицасының нормасынан алынған меншіксіз интегралдар шексіз деп ұйғарылған.

Түйінді сөздер: жай дифференциалдық теңдеулер, сингулярлық шекаралық есеп, шектелген шешім, аппроксимациялар, ерекше нүктелердегі шешімдердің әрекеті, параметрлеу әдісі

Кокотова Е.В.
Ограниченные решения дифференциальных систем с сингулярностями и их аппроксимации

Аннотация. Рассматриваются сингулярные краевые задачи для линейной неоднородной системы обыкновенных дифференциальных уравнений на конечном интервале. Предполагается, что на полуинтервалах несобственные интегралы от нормы матрицы коэффициентов бесконечны.

Ключевые слова: обыкновенные дифференциальные уравнения, сингулярная краевая задача, ограниченное решение, аппроксимации, поведение решений в особых точках, метод параметризации.

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AN APPROACH TO SOLVING A NONLINEAR BOUNDARY VALUE PROBLEM FOR A FREDHOLM INTEGRO-DIFFERENTIAL EQUATION

Summary. *A nonlinear boundary value problem for a Fredholm integro-differential equation is considered. The interval where the problem is considered is partitioned and the values of a solution to the problem at the left endpoints of the subintervals are introduced as additional parameters. The introduction of additional parameters gives initial values at the left endpoints of subintervals for new unknown functions. The* *considered integro-differential equation is reduced to a special Cauchy problem with parameters for a system of integro-differential equations. If this problem is solvable, then its solution can be represented using the introduced parameters and known values of the integro-differential equation. By substituting these representations into the boundary condition and the continuity conditions of the solution at the interior partition points, a system of nonlinear algebraic equations in the introduced parameters is constructed. The solvability of the boundary value problem is reduced to that of the system of algebraic equations. The conditions for the existence of a solution to the auxiliary system of algebraic equations are established.*

Key words: *nonlinear boundary value problem for the Fredholm integro-differential equation, special Cauchy problem, Dzhumabaev parameterization method.*

Introduction

Integro-differential equations often occur in various fields of natural science as mathematical models of real processes. As a rule, these processes are governed by nonlinear laws and, consequently, are described by nonlinear equations. The non-linearity of integro-differential equations leads to fundamental difficulties in solving problems for these equations and in establishing their qualitative properties.

Various initial and boundary value problems for integro-differential equations have been studied by many authors. They developed qualitative research methods, approximate and numerical methods for finding solutions to problems for integro-differential equations. Fredholm integro-differential equations have a number of features that should be taken into account when setting problems for these equations and developing methods for solving them.

In particular, as shown in [1, 2], a linear inhomogeneous Fredholm integro-differential equation can be unsolvable without additional conditions to the solution. Note that the criteria for solvability and unique solvability of linear boundary value problems for Fredholm integro-differential equations were obtained relatively recently [3]. In [4], the solvability conditions for the linear Fredholm integro-differential equation and boundary value problems for this equation are established.

Nonlinear problems are mainly solved by iterative methods. In this paper, we study the solvability of a nonlinear boundary value problem for the Fredholm integro-differential equation using the Dzhumabaev parametrization method [5].

Methods and results

We consider the nonlinear boundary value problem for the Fredholm integro-differential equation

$$\frac{dx}{dt} = f_0(t,x) + \int_0^T f_1\big(t,\tau,x(\tau)\big)d\tau, \quad t \in [0,T], \quad x \in \mathbb{R}^n. \tag{1}$$

$$g[x(0),x(T)] = 0, \tag{2}$$

where $f_0: [0,T] \times R^n \to R^n$, $f_1: [0,T] \times [0,T] \times R^n \to R^n$, $g: R^n \times R^n \to R^n$ are continuous, $\|x\| = \max_{i=\overline{1,n}}|x_i|$.

Let $C([0,T],R^n)$ denote the space of continuous functions $x: [0,T] \to R^n$ with the norm $\|x(t)\|_1 = \max_{t\in[0,T]}\|x(t)\|$.

The solution to problems (1) and (2) is continuously differentiable on $[0,T]$ vector function $x(t) \in C([0,T],\mathbb{R}^n)$ that satisfies the system of integro-differential equations (1) (at the points t=0, t=T, the system (1) is satisfied by one-sided derivatives $\dot{x}_{right}(0)$, $\dot{x}_{left}(T)$) and having at $t = 0$, $t = T$ the values $x(0)$, $x(T)$, for which equality (2) is true.

We divide the interval $[0,T)$ into N parts: $[0,T) = \cup_{r=1}^{N}[(r-1)h, rh)$ with step size $h > 0: Nh = T$ $(N = 1,2,\ldots)$. Let $x_r(t)$ denote the restriction of the function $x(t)$ on the rth subinterval $[(r-1)h, rh)$, i.e. $x_r(t) = x(t)$ for $t \in [(r-1)h, rh)$. Problems (1) and (2) are reduced to the equivalent multipoint problem

$$\frac{dx_r}{dt} = f_0(t, x_r) + \sum_{j=1}^{N} \int_{(j-1)h}^{jh} f_1\left(t, \tau, x_j(\tau)\right) d\tau, \qquad t \in [(r-1)h, rh), \qquad r = \overline{1, N}. \tag{3}$$

$$g\left[x_1(0), \lim_{t \to T-0} x_N(t)\right] = 0, \tag{4}$$

$$\lim_{t \to sh-0} x_s(t) = x_{s+1}(sh), s = \overline{1, N-1}, \tag{5}$$

where (5) are the continuity conditions of a solution at the interior partition points.

We denote by $C([0,T], h, R^{nN})$ the space of function systems $x[t] = \left(x_1(t), x_2(t), \dots, x_N(t)\right)$, where $x_r: [(r-1)h, rh) \to R^n$ are continuous functions having finite left-sided limits $\lim_{t \to rh-0} x_r(t)$, $r = \overline{1, N}$, with the norm $\|x[\cdot]\|_2 = \max_{r=\overline{1,N}} sup_{t \in [(r-1)h, rh)} \|x_r(t)\|$.

If $x^*(t)$ is a solution to boundary value problems (1) and (2), then the system of its restrictions $x^*[t] = \left(x_1^*(t), x_2^*(t), \dots, x_N^*(t)\right)$ is a solution to multipoint boundary value problem (3)-(5). Conversely, if a function system of $\tilde{x}[t] = \left(\tilde{x}_1(t), \tilde{x}_2(t), \dots, \tilde{x}_N(t)\right)$ is a solution to problems (3)-(5), then the function $\tilde{x}(t)$ obtained by splicing the function system is a solution to boundary value problems (1) and (2).

By introducing parameters $\lambda_r = x_r\left((r-1)h\right)$ and substituting $u_r(t) = x_r(t) - \lambda_r$, $r = \overline{1, N}$, on each r th subinterval, we get the multipoint boundary value problem with parameters

$$\frac{du_r}{dt} = f_0(t, u_r + \lambda_r) + \sum_{j=1}^{N} \int_{(j-1)h}^{jh} f_1(t, \tau, u_j(\tau) + \lambda_j) d\tau, \qquad t \in [(r-1)h, rh), \tag{6}$$

$$u_r\left((r-1)h\right) = 0, \qquad r = \overline{1, N}, \tag{7}$$

$$g\left[\lambda_1, \lambda_N + \lim_{t \to T-0} u_N(t)\right] = 0, \tag{8}$$

$$\lambda_s + \lim_{t \to sh-0} u_s(t) = \lambda_{s+1}, \qquad s = \overline{1, N-1}. \tag{9}$$

A pair $(\lambda^*, u^*[t])$ with elements $\lambda^* = (\lambda_1^*, \lambda_2^*, \dots, \lambda_N^*) \in R^{nN}$, $u^*[t] = \left(u_1^*(t), u_2^*(t), \dots, u_N^*(t)\right) \in C([0,T], h, R^{nN})$ is called a solution to problems (6) to (9) if the function $u_r^*(t)$ is continuously differentiable on $[(r-1)h, rh)$, $r = \overline{1, N}$, and, for $\lambda = \lambda^*$, satisfies systems (6), (8) and (9) and initial condition (7).

In [6], sufficient conditions for the existence of a unique solution to special Cauchy problems (6) and (7) and an estimate of the difference of its solutions corresponding to different values of the parameter were obtained. By using the algorithm proposed in [7] for fixed values of $\hat{\lambda} \in R^{nN}$, the system of functions $u[t, \hat{\lambda}]$ can be determined from the special Cauchy problem for the system of integro-differential equations (6) and (7).

If $\tilde{x}(t)$ is a solution to problems (1) and (2), then we compose the vector $\tilde{\lambda} = \left(\tilde{x}(0), \tilde{x}(h), \dots, \tilde{x}((N-1)h)\right) \in R^{nN}$ and the function system $\tilde{u}[t] = \left(\tilde{u}_1(t), \tilde{u}_2(t), \dots, \tilde{u}_N(t)\right)$, where $\tilde{u}_r(t)$ is the restriction of the function $\tilde{x}(t) - \tilde{x}\left((r-1)h\right)$ on the rth interval. It is obvious that $\tilde{u}[t] \in C([0,T], h, R^{nN})$ and the pair $(\tilde{\lambda}, \tilde{u}[t])$ is a solution to problems (6)-(9). And vice versa, if $u^*[t] = u[t, \lambda^*]$ and the pair $(\lambda^*, u^*[t])$ is a solution to problems (6)-(9), the function $x^*(t)$, defined by the equalities

$x^*(t) = \lambda_r^* + u_r^*(t)$, $t \in [(r-1)h, rh)$, $r = \overline{1,N}$ and $x^*(T) = \lambda_N^* + \lim_{t\to T-0} u_N^*(t)$, is a solution to the original problems (1) and (2).

For a fixed value of the parameter $\lambda \in R^{nN}$, the problems (6) and (7) is equivalent to the system of Volterra integral equations of the second kind

$$u_r(t) = \int_{(r-1)h}^{t} \sum_{j=1}^{N} \int_{(j-1)h}^{jh} f_1(\tau_1, \tau, u_j(\tau) + \lambda_j) d\tau d\tau_1 + \int_{(r-1)h}^{t} f_0(\tau_1, u_r(\tau_1) + \lambda_r) d\tau_1, \quad t \in [(r-1)h, rh), \quad r = \overline{1,N}. \tag{10}$$

Defining $\lim_{t\to rh-0} x_r(t)$, $r = \overline{1,N}$, from (10) and substituting them in (8) and (9), we obtain the system of nonlinear algebraic equations in the parameters λ:

$$g\left[\lambda_1, \lambda_N + \int_{(N-1)h}^{Nh} \sum_{j=1}^{N} \int_{(j-1)h}^{jh} f_1(\tau_1, \tau, u_j(\tau) + \lambda_j) d\tau d\tau_1 + \int_{(N-1)h}^{Nh} f_0(\tau_1, u_N(\tau_1) + \lambda_N) d\tau_1\right] = 0, \tag{11}$$

$$\lambda_s + \int_{(s-1)h}^{sh} \sum_{j=1}^{N} \int_{(j-1)h}^{jh} f_1(\tau_1, \tau, u_j(\tau) + \lambda_j) d\tau d\tau_1 + \int_{(s-1)h}^{sh} f_0(\tau_1, u_N(\tau_1) + \lambda_N) d\tau_1 - \lambda_{s+1} = 0, \quad s = \overline{1, N-1}. \tag{12}$$

We write the system of equations (10) and (11) as follows:

$$Q_*(h; \lambda) = 0, \quad \lambda \in \mathbb{R}^{nN}. \tag{13}$$

Given a vector $\lambda^{(0)} = (\lambda_1^{(0)}, \lambda_2^{(0)}, \dots, \lambda_N^{(0)}) \in R^{nN}$, we define the piecewise constant vector-function $x^{(0)}(t)$ by the equalities $x^{(0)}(t) = \lambda_r^{(0)}$, $t \in [(r-1)h, rh)$, $r = \overline{1,N}$, $x^{(0)}(T) = \lambda_N^{(0)}$.

Let $PC([0,T], h, R^n)$ denote the space of piecewise continuous functions $x(t): [0,T] \to R^n$ with the possible discontinuity points $t_i = (i-1)h$, $i = \overline{1, N-1}$, with the norm $\|x\|_3 = \sup_{t\in[0,T]} \|x(t)\|$.

Taking some numbers $\rho_\lambda > 0, \rho > \rho_\lambda$, we compose the sets

$$G_0(\rho) = \{(t,x): t \in [0,T], \|x - x^{(0)}(t)\| < \rho\},$$

$$G_1(\rho) = \{(t,s,x): t \in [0,T], s \in [0,T], \|x - x^{(0)}(s)\| < \rho\},$$

$$S(\lambda^{(0)}, \rho_\lambda) = \{\lambda \in R^{nN}: \|\lambda - \lambda^{(0)}\| < \rho_\lambda\},$$

$$S(x^{(0)}(t), \rho) = \{x(t) \in \mathbb{PC}([0,T], h, \mathbb{R}^n): \|x - x_0\|_3 < \rho\},$$

$$S(0, \rho_u) = \{u[t] \in C([0,T], h, R^{nN}): \|u[\cdot]\|_2 < \rho_u\}, \rho_u \le \rho - \rho_\lambda.$$

Condition A. The functions $f_0(t,x)$, $f_1(t,s,x)$ *are continuous in* $G_0(\rho)$, $G_1(\rho)$, *respectively, have continuous partial derivatives* $\frac{\partial f_0(t,x)}{\partial x}$, $\frac{\partial f_1(t,s,x)}{\partial x}$ *and satisfy the inequalities* $\left\|\frac{\partial f_0(t,x)}{\partial x}\right\| \le L_0$, $(t,x) \in G_0(\rho)$, $\left\|\frac{\partial f_1(t,s,x)}{\partial x}\right\| \le L_1$, $(t,s,x) \in G_1(\rho)$.

Condition B. The function $g(v,w)$ *has uniformly continuous partial derivatives* $g_v'(v,w)$ *and* $g_w'(v,w)$ *in* $G_2(\rho,\rho) = \{(v,w) \in \mathbb{R}^{2n}: \|v - x^{(0)}(0)\| < \rho, \|w - x^{(0)}(T)\| < \rho\}$.

Theorem 1. Let $\lambda^* \in S(\lambda^{(0)}, \rho_\lambda)$ *be a solution to equation (13) and* $u^*[t] \in S(0, \rho_u)$ *be a solution to special Cauchy problems (6) and (7) for* $\lambda = \lambda^*$. *Then the function* $x^*(t)$, *defined by the equalities* $x^*(t) = \lambda_r^* + u_r^*(t)$, $t \in [(r-1)h, rh)$, $r = \overline{1,N}$, *and* $x^*(T) = \lambda_N^* + \lim_{t\to T-0} u_N^*(t)$, *is a solution to problems (1) and (2) and* $x^*(t) \in S(x^{(0)}(t), \rho)$.

To solve the system of nonlinear algebraic equations (13), we use the following statement.

Theorem 2. Let the following conditions be fulfilled:

(i) *the Jacobi matrix* $\frac{\partial Q_*(\Delta_N;\lambda)}{\partial \lambda}$ *is uniformly continuous in* $S(\lambda^{(0)}, \rho_\lambda)$;

(ii) $\frac{\partial Q_*(\Delta_N;\lambda)}{\partial \lambda}$ *is invertible and* $\left\| \left[\frac{\partial Q_*(\Delta_N;\lambda)}{\partial \lambda} \right]^{-1} \right\| \le \gamma^*$ *for all* $\lambda \in S(\lambda^{(0)}, \rho_\lambda)$, γ^* *is const;*

(iii) $\gamma^* \| Q_*(\Delta_N; \lambda^{(0)}) \| < \rho_\lambda$.

Then there exists $\alpha_0 \ge 1$ *such that for any* $\alpha \ge \alpha_0$ *the sequence* $\{\lambda^{(k+1)}\}$, *generated by the iterative process*

$$\lambda^{(k+1)} = \lambda^{(k)} - \frac{1}{\alpha}\left[\frac{\partial Q_*(\Delta_N;\lambda^{(k)})}{\partial \lambda}\right]^{-1} Q_*(\Delta_N;\lambda^{(k)}), \quad k = 0, 1, 2, \ldots, \tag{14}$$

converges to λ^*, *an isolated solution to equation (14) in* $S(\lambda^{(0)}, \rho_\lambda)$, *and*

$$\|\lambda^* - \lambda^{(0)}\| \le \gamma^* \|Q_*(\Delta_N;\lambda^{(0)})\|. \tag{15}$$

Conclusion

In this paper, we proposed an approach to solving a nonlinear boundary value problem for the Fredholm integro-differential equation. This approach is based on the parameterization method.

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Мынбаева С.Т., Каракенова С.Г.

Фредгольм интегралдық-дифференциалдық теңдеуі үшін сызықты емес шеттік есепті шешудің бір тәсілі туралы

Аңдатпа: Фредгольм интегралдық-дифференциалдық теңдеуі үшін сызықты емес шеттік есеп қарастырылады. Есеп қарастырылатын аралық бөліктерге бөлінеді және есеп шешімінің ішкі аралықтардың бастапқы нүктелеріндегі мәндері қосымша параметрлер ретінде енгізіледі. Қосымша параметрлерді енгізу жаңа белгісіз функциялар үшін ішкі аралықтардың сол жақ шеткі нүктелерінде бастапқы мәндерді береді. Зерттеліп отырған интегралдық-дифференциалдық теңдеу интегралдық-дифференциалдық теңдеулер жүйесі үшін параметрлері бар арнайы Коши есебіне келтіріледі. Егер бұл есеп шешілімді болса, онда оның шешімін енгізілген параметрлер мен интегралдық-дифференциалдық теңдеудің белгілі шамалары арқылы өрнектеуге болады. Осы өрнектерді шеттік шартқа және бөліктеудің ішкі нүктелеріндегі шешімнің үзіліссіздік шарттарына қоя отырып, енгізілген параметрлерге қатысты сызықты емес алгебралық теңдеулер жүйесі құрылады. Шеттік есептің шешілімділігі алгебралық теңдеулер жүйесінің шешімділігіне келтіріледі. Көмекші алгебралық теңдеулер жүйесінің шешімінің бар болу шарттары алынды.

Түйінді сөздер: Фредгольм интегралдық-дифференциалдық теңдеуі үшін сызықты емес шеттік есеп, арнайы Коши есебі, Жұмабаевтың параметрлеу әдісі

Мынбаева С.Т., Каракенова С.Г.

Об одном подходе к решению нелинейной краевой задачи для интегро-дифференциального уравнения Фредгольма

Аннотация. Рассматривается нелинейная краевая задача для интегро-дифференциального уравнения Фредгольма. Интервал где рассматривается задача делится на части и значения решения в начальных точках подинтервалов вводятся в качестве дополнительных параметров. Введение дополнительных параметров дает начальные условия в левых точках подинтервалов для новых неизвестных функций. Рассматриваемое интегро-дифференциальное уравнение сводится к специальной задаче Коши с параметрами для системы интегро-дифференциальных уравнений. Если эта задача разрешима, то ее решение можно представить с помощью введенных параметров и известных величин интегро-дифференциального уравнения. Подставляя представление решения в краевое условие при соответствующих значениях и условия непрерывности решения во внутренних точках разбиения, составляется система нелинейных алгебраических уравнений относительно введенных параметров. Разрешимость нелинейной краевой задачи сводится к разрешимости системы алгебраических уравнений. Установлены условия существования решения вспомогательной системы нелинейных алгебраических уравнений.

Ключевые слова: нелинейная краевая задача для интегро-дифференциального уравнения Фредгольма, специальная задача Коши, метод параметризации Джумабаева.

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