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A MODIFICATION OF ALGORITHMS OF THE DZHUMABAEV PARAMETERIZATION METHOD AND A NUMERICAL METHOD

Abstract. *The article deals with a modification of the algorithms of D. S. Dzhumabaev's parametrization method. Additional parameters are introduced at the internal partition points and at both ends of the interval. Sufficient conditions for convergence of these algorithms in terms of input data are given. Using the right-hand part of the system of differential equations and the boundary condition function, a nonlinear operator equation was constructed to find initial approximations of unknown parameters. A numerical method is proposed for finding a solution to a nonlinear two-point boundary value problem for a system of ordinary differential equations. The numerical method was implemented in a test example.*

Key words: *nonlinear two-point boundary value problem, the Dzhumabaev parametrization method, sufficient conditions, an isolated solution, numerical method.*

In Memory of Professor Dulat S. Dzhumabaev

We consider a nonlinear two-point boundary value problem for a system of ordinary differential equations

$$\frac{dx}{dt} = f(t,x), \quad t \in [0,T], \quad x \in R^n, \tag{1}$$

$$g(x(0), x(T)) = 0, \tag{2}$$

where $f:[0,T]\times R^n \to R^n$ and $g: R^n \times R^n \to R^n$ are continuous and $\|x\| = \max_{i=\overline{1,n}} |x_i|$.

By $C([0,T],R^n)$ we denote a space of continuous functions $x:[0,T]\to R^n$ with norm $\|x\|_1 = \max_{t\in[0,T]} \|x(t)\|$.

Problems of solvability and construction of approximate methods for the solution of problems (1) and (2) are studied in numerous papers [1-11]. For the bibliography and detailed analysis of the works dealing with the main groups of methods aimed at the investigation and solution of boundary-value problems, see [11].

The main goal of the work is using one modification of the algorithms of the Dzhumabaev parametrization method [12-14], to establish sufficient conditions for the existence of an isolated solution of boundary value problems (1) and (2), and to propose a numerical method for solving boundary value problems (1) and (2).

For the chosen points $\Delta_N : 0 = t_0 < t_1 < t_2 < \ldots < t_N = T$, where $N = 1,2,\ldots$, we perform the partition $[0,T) = \bigcup_{r=1}^{N}[t_{r-1},t_r)$. Denote by $x_r(t)$ the function $x(t)$ restricted to the r th interval $[t_{r-1},t_r)$. By $C([0,T],\Delta_N,R^{nN})$ we denote a space of the systems of functions $x[t] = (x_1(t), x_2(t),\ldots,x_N(t))$, where $x_r:[t_{r-1},t_r)\to R^n$ are continuous and have finite left limits $\lim_{t\to t_r-0} x_r(t)$ for all $r=\overline{1,N}$ with the norm $\|x[\cdot]\|_2 = \max_{r=\overline{1,N}} \sup_{t\in[t_{r-1},t_r)} \|x_r(t)\|$. It is clear that $C([0,T],\Delta_N,R^{nN})$ is a complete space.

Denote by λ_r the value of $x_r(t)$ at $t = t_{r-1}$ $(r=\overline{1,N})$ and λ_{N+1} the limit $\lim\limits_{t\to t_N} x_N(t)$. Here in contrast to the classical parameterization method the parameter is also entered at the point $t = T$, so we will use notations $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N, \lambda_{N+1}) \in R^{n(N+1)}$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N) \in R^{nN}$. We change the variable according to $u_r(t) = x_r(t) - \lambda_r$ on each interval $[t_{r-1}, t_r)$. Then, we obtain the boundary value problem with parameters

$$\frac{du_r}{dt} = f(t, u_r + \lambda_r), \quad t \in [t_{r-1}, t_r), \quad r = \overline{1,N}, \tag{3}$$

$$u_r(t_{r-1}) = 0, \qquad r = \overline{1,N}, \tag{4}$$

$$g(\lambda_1, \lambda_{N+1}) = 0, \tag{5}$$

$$\lambda_r + \lim_{t\to t_r - 0} u_r(t) - \lambda_{r+1} = 0, \quad r = \overline{1,N}. \tag{6}$$

Let $x(t)$ be a solution to problems (1) and (2). Then, the system of pairs ($\lambda_r = x_r(t_{r-1})$, $u_r(t) = x_r(t) - x_r(t_{r-1})$), $r = \overline{1,N}$, is a solution to problems (3)-(6) (here, $x_r(t)$ is a restriction of $x(t)$ on $[t_{r-1}, t_r)$). Conversely, if $(\tilde{\lambda}_r, \tilde{u}_r(t))$, $r = \overline{1,N}$, is a solution to (3)-(6), then the function $\tilde{x}(t)$ defined by the equalities $\tilde{x}(t) = \begin{cases} \tilde{\lambda}_r + \tilde{u}_r(t), & for\ t \in [t_{r-1}, t_r), \quad r = \overline{1,N}, \\ \tilde{\lambda}_{N+1}, & for\ t = t_N, \end{cases}$ is a solution to problems (1) and (2). What makes the parametric problem advantageous over (3)-(6) is the presence of the initial conditions $u_r(t_{r-1}) = 0$, $r = \overline{1,N}$. For a fixed λ_r, Cauchy problems (3) and (4) are equivalent to the Volterra nonlinear integral equation

$$u_r(t) = \int_{t_{r-1}}^{t} f(\tau, \lambda_r + u_r(\tau))d\tau, \quad t \in [t_{r-1}, t_r), \quad r = \overline{1,N}. \tag{7}$$

Replace $u_r(\tau)$ on the right-hand side of (7) by its integral representation. Then, $u_r(t)$ can be written as

$$u_r(t) = \int_{t_{r-1}}^{t} f\left(\tau_1, \lambda_r + \int_{t_{r-1}}^{\tau_1} f(\tau_2, \lambda_r + u_r(\tau_2))d\tau_2\right)d\tau_1, \quad t \in [t_{r-1}, t_r), \quad r = \overline{1,N}.$$

From this relation, we find $\lim\limits_{t\to t_r - 0} u_r(t)$ $(r = \overline{1,N})$. Substituting these values into (5) and (6), we obtain the following system of nonlinear equations for $\lambda_r \in R^n$:

$$g(\lambda_1, \lambda_{N+1}) = 0,$$

$$\lambda_r + \int_{t_{r-1}}^{t_r} f\left(\tau_1, \lambda_r + \int_{t_{r-1}}^{\tau_1} f(\tau_2, \lambda_r + u_r(\tau_2))d\tau_2\right)d\tau_1 - \lambda_{r+1} = 0, \quad r = \overline{1,N}.$$

We write this system in the form

$$Q_{2,\Delta_N}(\Lambda, u) = 0, \quad \Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N, \lambda_{N+1}) \in R^{n(N+1)}. \tag{8}$$

Condition A. *There exists* Δ_N *such that the system of nonlinear equations* $Q_{2,\Delta_N}(\Lambda, 0) = 0$ *has a solution* $\Lambda^{(0)} = \left(\lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_N^{(0)}, \lambda_{N+1}^{(0)}\right) \in R^{n(N+1)}$, *for* $\lambda^{(0)} = \left(\lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_N^{(0)}\right) \in R^{nN}$ *the Cauchy problem*

$$\frac{du_r}{dt} = f\left(t, \lambda_r^{(0)} + u_r\right), \quad u_r(t_{r-1}) = 0, \quad t \in [t_{r-1}, t_r),$$

has a solution $u_r^{(0)}(t)$ *for all* $r = \overline{1,N}$, *and the system of functions is such that*

$$u^{(0)}[t] = \left(u_1^{(0)}(t), u_2^{(0)}(t), \ldots, u_N^{(0)}(t)\right) \in C([0,T], \Delta_N, R^{nN})$$

We define the function $x^{(0)}(t)$ as follows:

$$x^{(0)}(t) = \begin{cases} \lambda_r^{(0)} + u_r^{(0)}(t), & for\ t \in [t_{r-1}, t_r), \quad r = \overline{1,N}, \\ \lambda_{N+1}^{(0)}, & for\ t = t_N. \end{cases}$$

We take the numbers $\rho_\lambda > 0, \rho_u > 0$, and $\rho_x > 0$ and compose the sets

$$S(\Lambda^{(0)},\rho_\lambda)=\left\{\Lambda\in R^{n(N+1)}:\left\|\Lambda-\Lambda^{(0)}\right\|=\max_{r=\overline{1,N+1}}\left\|\lambda_r-\lambda_r^{(0)}\right\|<\rho_\lambda\right\},$$

$$S(\lambda^{(0)},\rho_\lambda)=\left\{\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_N)\in R^{nN}:\left\|\lambda-\lambda^{(0)}\right\|=\max_{r=\overline{1,N}}\left\|\lambda_r-\lambda_r^{(0)}\right\|<\rho_\lambda\right\},$$

$$S(u^{(0)}[t],\rho_u)=\left\{u[t]\in C([0,T],\Delta_N,R^{nN}):\left\|u[\cdot]-u^{(0)}[\cdot]\right\|_2<\rho_u\right\},$$

$$G_1^0(\rho_x)=\left\{(t,x):t\in[0,T],\left\|x-x^{(0)}(t)\right\|<\rho_x\right\},$$

$$G_2^0(\rho_\lambda,\rho_x)=\left\{(v,w)\in R^{2n}:\left\|v-\lambda_1^{(0)}\right\|<\rho_\lambda,\left\|w-\lambda_{N+1}^{(0)}\right\|<\rho_x\right\}.$$

Condition B**.** *The functions f and g have uniformly continuous partial derivatives f_x' in $G_1^0(\rho_x)$, and g_v' in $G_2^0(\rho_\lambda,\rho_x)$, and g_w' in $G_2^0(\rho_\lambda,\rho_x)$, and the following inequalities are true:*

$$\left\|f_x'(t,x)\right\|\leq L,\quad \left\|g_v'(v,w)\right\|\leq L_1,\quad \left\|g_w'(v,w)\right\|\leq L_2,$$

where L and L_1 and L_2 are constants.

Let Condition A be met. We take the system of pairs $(\lambda_r^{(0)},u_r^{(0)}(t))$, $r=\overline{1,N}$. Let's define the sequence of pairs $(\lambda_r^{(k)},u_r^{(k)}(t))$, $r=\overline{1,N}$, by the following algorithm.

Step 1. (a) Determine the parameter $\Lambda^{(1)}=(\lambda_1^{(1)},\ldots,\lambda_N^{(1)},\lambda_{N+1}^{(1)})\in R^{n(N+1)}$ from Eq. (8) with $u=u^{(0)}$; (b) Find $u_r^{(1)}(t)$, $t\in[t_{r-1},t_r)$, solving Cauchy problems (3) and (4) with $\lambda_r=\lambda_r^{(1)}$, $r=\overline{1,N}$.

Step 2. (a) Replacing u by the calculated function $u^{(1)}$ and solving Eq. (8), determine $\Lambda^{(2)}\in R^{n(N+1)}$; (b) Find $u_r^{(2)}(t)$, $t\in[t_{r-1},t_r)$, solving Cauchy problems (3) and (4) with $\lambda_r=\lambda_r^{(2)}$, $r=\overline{1,N}$.

Continuing this process, at the k th step of the algorithm, we obtain the system of pairs $(\lambda_r^{(k)},u_r^{(k)}(t))$, $r=\overline{1,N}$. Using $(\lambda_r^{(k)},u_r^{(k)}(t))$, $r=\overline{1,N}$, form the pair $(\lambda^{(k)},u^{(k)}[t])$, where $\lambda^{(k)}=(\lambda_1^{(k)},\ldots,\lambda_N^{(k)})$ and $u^{(k)}[t]=(u_1^{(k)}(t),\ldots,u_N^{(k)}(t))$.

The algorithms of the parameterization method open up the prospect for further development of constructive methods that allow simultaneous investigation of the existence and construction of solutions to boundary value problems for differential equations.

Theorem. 1*Assume that there exist Δ_N, $\rho_\lambda>0$, $\rho_u>0$ and $\rho_x>0$ for which Conditions A and B are satisfied, the $(n(N+1)\times n(N+1))$ Jacobi matrix $\dfrac{\partial Q_{2,\Delta_N}(\Lambda,u)}{\partial\Lambda}:R^{n(N+1)}\times R^{n(N+1)}$ is invertible for all $\Lambda\in S(\Lambda^{(0)},\rho_\lambda)$ and $u[t]\in S(u^{(0)}[t],\rho_u)$ and the following inequalities are true:*

1) $\left\|\left(\dfrac{\partial Q_{2,\Delta_N}(\Lambda,u)}{\partial\Lambda}\right)^{-1}\right\|\leq\gamma_2(\Delta_N)$, *where $\gamma_2(\Delta_N)$ is constant,*

2) $q_2(\Delta_N)=\gamma_2(\Delta_N)\max\limits_{r=\overline{1,N}}\left\{\exp(L(t_r-t_{r-1}))-\sum\limits_{j=0}^{2}\dfrac{(L(t_r-t_{r-1}))^j}{j!}\right\}<1,$

3) $\dfrac{\gamma_2(\Delta_N)}{1-q_2(\Delta_N)}\left\|Q_{2,\Delta_N}(\Lambda^{(0)},u^{(0)})\right\|<\rho_\lambda,$

4) $\dfrac{\gamma_2(\Delta_N)}{1-q_2(\Delta_N)}\max\limits_{r=\overline{1,N}}\left\{\exp(L(t_r-t_{r-1}))-1\right\}\left\|Q_{2,\Delta_N}(\Lambda^{(0)},u^{(0)})\right\|<\rho_u,$

5) $\max\limits_{p=1,2}\left\{\rho_\lambda\max\limits_{r=\overline{1,N}}\sum_{j=0}^{p-1}\dfrac{(L(t_r-t_{r-1}))^j}{j!}+\rho_u\max\limits_{r=\overline{1,N}}\dfrac{(L(t_r-t_{r-1}))^{p-1}}{(p-1)!}\right\}\leq\rho_x.$

Then, the sequence of pairs $\left(\lambda^{(k)}, u^{(k)}[t]\right)$, $k=0,1,\dots$, *is contained in* $S\left(\lambda^{(0)},\rho_\lambda\right)\times S\left(u^{(0)}[t],\rho_u\right)$*and converges to* $\left(\lambda^*, u^*[t]\right)$, *which is a solution to problem (3)-(6) belonging to* $S\left(\lambda^{(0)},\rho_\lambda\right)\times S\left(u^{(0)}[t],\rho_u\right)$. *Furthermore, the following estimates are valid:*

(a) $\left\|\Lambda^*-\Lambda^{(k)}\right\| \le (q_2(\Delta_N))^k \dfrac{\gamma_2(\Delta_N)}{1-q_2(\Delta_N)}\left\|Q_{2,\Delta_N}\left(\Lambda^{(0)},u^{(0)}\right)\right\|$, $k=0,1,2,\dots,$

(b) $\left\|u_r^*(t)-u_r^{(k)}(t)\right\| \le \left(\exp(L(t-t_{r-1}))-1\right)\left\|\lambda_r^*-\lambda_r^{(k)}\right\|$, $t\in[t_{r-1},t_r)$, $r=\overline{1,N}$.

Moreover, any solution to problem (3)-(6) belonging to $S\left(\lambda^{(0)},\rho_\lambda\right)\times S\left(u^{(0)}[t],\rho_u\right)$ *is isolated.*

Based on the proved Theorem, we propose a numerical method for solving problems (1) and (2):

1) A system of equations $Q_{2,\Delta_N}(\lambda,0)=0$ is compiled.

2) Some vector $\lambda^{(0,0)}\in R^{2(N+1)}$ is selected and an iterative process [13] is constructed to find the solution of the $\lambda^{(0)}$ equation $Q_{2,\Delta_N}(\lambda,0)=0$:

$$\lambda^{(0,m+1)}=\lambda^{(0,m)}-\frac{1}{\alpha}\left(\frac{\partial Q_{2,\Delta_N}(\lambda^{(0,m)},0)}{\partial\lambda}\right)^{-1}Q_{2,\Delta_N}(\lambda^{(0,m)},0),\quad m=0,1,2,\dots.$$

3) The solution to the Cauchy problem is found

$$\frac{dx_r}{dt}=f(t,x_r),\quad t\in[t_{r-1},t_r),\quad x_r\in R^n,\quad r=\overline{1,N},$$

$$x_r(t_{r-1})=\lambda_r^{(0)},\quad r=\overline{1,N}.$$

4) According to Theorem 1, the piecewise continuous function

$$x^{(0)}(t)=\begin{cases} x_r^{(0)}(t), & t\in[t_{r-1},t_r),\ r=\overline{1,N},\\ \lambda_{N+1}^{(0)}, & t=t_N,\end{cases}$$

is an approximate solution of the problem (1), (2) with an error not exceeding

$$\varepsilon=\left(\max_{r=\overline{1,N}}\sup_{t\in[t_{r-1},t_r)}\left\{\exp(L(t-t_{r-1}))-1\right\}+1\right)\frac{\gamma_2(\Delta_N)}{1-q_2(\Delta_N)}\left\|Q_{2,\Delta_N}(\lambda^{(0)},u^{(0)})\right\|.$$

Example.

We consider a nonlinear two point boundary value problem for a second order ordinary differential equation [15]

$$y''=y^2+2\pi^2\cos 2\pi t-\sin^4\pi t,\quad t\in(\pi/6,11\pi/24),\quad y\in R, \tag{9}$$

$$\begin{cases}\operatorname{atan}\left(y\left(\frac{\pi}{6}\right)\right)=0.782647,\\ \exp\left(\frac{1}{2}y'\left(\frac{\pi}{6}\right)\right)-\exp\left(-\frac{1}{2}y^2\left(\frac{11\pi}{24}\right)\right)=0.165009.\end{cases} \tag{10}$$

We will perform a replacement: $y=x_1$, $y'=x_2$. We will get the boundary value problem for the system of ordinary diffferential equations

$$\begin{cases}\dfrac{dx_1}{dt}=x_2,\\ \dfrac{dx_2}{dt}=x_1^2+2\pi^2\cos 2\pi t-\sin^4\pi t,\end{cases}\quad t\in(\pi/6,11\pi/24),\quad x_1,x_2\in R, \tag{11}$$

$$\begin{cases}\operatorname{atan}\left(x_1\left(\frac{\pi}{6}\right)\right)=0.782647,\\ \exp\left(\frac{1}{2}x_2\left(\frac{\pi}{6}\right)\right)-\exp\left(-\frac{1}{2}x_1^2\left(\frac{11\pi}{24}\right)\right)=0.165009.\end{cases} \tag{12}$$

For the chosen points $\Delta_8 : t_r = \frac{\pi}{6} + \frac{r}{8}\left(\frac{11\pi}{24} - \frac{\pi}{6}\right)$, $r = \overline{0,8}$, we perform the partition $[\pi/6, 11\pi/24) = \bigcup_{r=1}^{8} [\pi(25+7r)/192, \pi(32+7r)/192)$. We will write down a system of nonlinear algebraic equations $Q_{2,\Delta_8}(\Lambda, 0) = 0$:

$$atan(\lambda_{1,1}) - 0.782647 = 0, \quad \exp\left(\frac{1}{2}\lambda_{1,2}\right) - \exp\left(-\frac{1}{2}\lambda_{9,1}^2\right) - 0.165009 = 0,$$

$$\frac{49\pi^2}{73728}\lambda_{r,1} + \frac{7\pi}{192}\lambda_{r,2} - \lambda_{r+1,1} + \frac{7}{6144}\sin\left(\frac{\pi^2(25+7r)}{48}\right) - \frac{7(1+4\pi^2)}{768}\sin\left(\frac{\pi^2(25+7r)}{96}\right) -$$

$$-\cos\left(\frac{\pi^2(32+7r)}{192}\right) + \cos\left(\frac{\pi^2(25+7r)}{192}\right) + \frac{1}{64\pi^2}\cos^2\left(\frac{\pi^2(32+7r)}{96}\right) - \frac{1}{4\pi^2}\cos^2\left(\frac{\pi^2(32+7r)}{192}\right) -$$

$$-\frac{1}{64\pi^2}\cos^2\left(\frac{\pi^2(25+7r)}{96}\right) + \frac{1}{4\pi^2}\cos^2\left(\frac{\pi^2(25+7r)}{192}\right) - \frac{49\pi^2}{196608} = 0, \quad r = \overline{1,8},$$

$$\frac{7\pi}{192}\lambda_{r,1} - \frac{49\pi^2}{36864}\lambda_{r,1}\lambda_{r,2} + \frac{343\pi^3}{21233664}\lambda_{r,2}^2 - \lambda_{r+1,2} -$$

$$-\frac{1}{32\pi}\sin\left(\frac{\pi^2(32+7r)}{48}\right) + \frac{1}{4\pi}\sin\left(\frac{\pi^2(32+7r)}{96}\right) +$$

$$+\frac{1}{32\pi}\sin\left(\frac{\pi^2(25+7r)}{48}\right) - \frac{1}{4\pi}\sin\left(\frac{\pi^2(25+7r)}{96}\right) +$$

$$+\pi\sin\left(\frac{\pi^2(32+7r)}{96}\right) - \pi\sin\left(\frac{\pi^2(25+7r)}{96}\right) - \frac{7\pi}{512} = 0, \quad r = \overline{1,8}.$$

We take the vector $\Lambda^{(0,0)} = \left(\begin{pmatrix}\lambda_{1,1}^{(0,0)}\\ \lambda_{1,2}^{(0,0)}\end{pmatrix}, \begin{pmatrix}\lambda_{2,1}^{(0,0)}\\ \lambda_{2,2}^{(0,0)}\end{pmatrix}, \ldots, \begin{pmatrix}\lambda_{9,1}^{(0,0)}\\ \lambda_{9,2}^{(0,0)}\end{pmatrix}\right) = \left(\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \ldots, \begin{pmatrix}1\\1\end{pmatrix}\right)$ as the initial approximation of the solution and find the solution to this system using Theorem 1 [13]. We construct the following iterative process:

$$\Lambda^{(0,m+1)} = \Lambda^{(0,m)} - \frac{1}{2}\left(\frac{\partial Q_{2,\Delta_8}(\Lambda^{(0,m)}, 0)}{\partial \Lambda}\right)^{-1} \cdot Q_{2,\Delta_8}(\Lambda^{(0,m)}, 0), \quad m = 0, 1, \ldots$$

Let's take $\Lambda^{(0,100)}$ as $\Lambda^{(0)}$, since $Q_{2,\Delta_8}(\Lambda^{(0,100)}, 0) = 0$:

$$\Lambda^{(0)} = \Lambda^{(0,100)} = \left(\begin{pmatrix}0.994513\\-0.469637\end{pmatrix}, \begin{pmatrix}0.823050\\-2.394092\end{pmatrix}, \begin{pmatrix}0.492786\\-3.135266\end{pmatrix}, \begin{pmatrix}0.166653\\-2.321885\end{pmatrix},\right.$$

$$\left.\begin{pmatrix}0.006523\\-0.353855\end{pmatrix}, \begin{pmatrix}0.092625\\1.792282\end{pmatrix}, \begin{pmatrix}0.382507\\3.050756\end{pmatrix}, \begin{pmatrix}0.732087\\2.800015\end{pmatrix}, \begin{pmatrix}0.968376\\1.168976\end{pmatrix}\right)$$

Now we will find solutions of Cauchy problems (at the points $t_{r,k} = \frac{\pi}{6} + (r-1)\frac{7\pi}{192} + (k-1)\frac{7\pi}{960}$, $r = \overline{1,8}$, $k = \overline{1,6}$):

$$\begin{cases} \dfrac{dx_{r,1}}{dt} = x_{r,2}, \\ \dfrac{dx_{r,2}}{dt} = x_{r,1}^2 + 2\pi^2\cos 2\pi t - \sin^4 \pi t, \end{cases}$$

$$t \in [\pi(25+7r)/192, \pi(32+7r)/192], \quad r = \overline{1,8},$$

$$x_{r,1}(\pi(25+7r)/192) = \Lambda_{r,1}^{(0)}, \quad x_{r,2}(\pi(25+7r)/192) = \Lambda_{r,2}^{(0)}, \quad r = \overline{1,8},$$

using the method of Runge-Kutta of fourth-order accuracy with a step $\frac{7\pi}{960} \approx 0.022907$. We will define the function

$$x_1^{(0)}(t) = \begin{cases} x_{r,1}^{(0)}(t), & t \in [\pi(25+7r)/192, \pi(32+7r)/192), \quad r = \overline{1,7}, \\ x_{8,1}^{(0)}(t), & t \in [433\pi/960, 11\pi/24]. \end{cases}$$

The exact solution to the boundary value problems (1) and (2) is the function $y(t) = \sin^2(\pi t)$, and the approximate solution is the function $x_1^{(0)}(t)$.

The following table shows the values of the numerical solution of the boundary value problems (1) and (2), as well as the difference between the numerical solution and the exact solution at the points $\hat{t}_j = \frac{\pi}{6} + (j-1)\frac{7\pi}{960}$, $j = \overline{1,41}$, where $\hat{t}_1 = \frac{\pi}{6} \approx 0.523599$, $\hat{t}_{41} = \frac{11\pi}{24} \approx 1.439897$:

Table 1

j	$\hat{t}_j$	$x_1^{(0)}(\hat{t}_j)$	$x_1^{(0)}(\hat{t}_j) - y(\hat{t}_j)$	j	$\hat{t}_j$	$x_1^{(0)}(\hat{t}_j)$	$x_1^{(0)}(\hat{t}_j) - y(\hat{t}_j)$
1	0,523599	0,994513	-0,000001	21	0,981748	0,006523	0,003239
2	0,546506	0,978678	-0,000127	22	1,004655	0,003582	0,003368
3	0,569414	0,952941	-0,000254	23	1,027563	0,010977	0,003498
4	0,592321	0,917831	-0,000381	24	1,050470	0,028557	0,003627
5	0,615229	0,874072	-0,000508	25	1,073377	0,055962	0,003756
6	0,638136	0,823050	-0,000152	26	1,096285	0,092625	0,003883
7	0,661043	0,765054	-0,000086	27	1,119192	0,137793	0,004009
8	0,683951	0,701576	-0,000018	28	1,142100	0,190535	0,004136
9	0,706858	0,633928	0,000048	29	1,165007	0,249764	0,004265
10	0,729766	0,563512	0,000115	30	1,187915	0,314257	0,004394
11	0,752673	0,492786	0,001184	31	1,210822	0,382507	0,004349
12	0,775581	0,421301	0,001320	32	1,233730	0,453412	0,004438
13	0,798488	0,351471	0,001455	33	1,256637	0,525374	0,004529
14	0,821396	0,284743	0,001591	34	1,279545	0,596908	0,004623
15	0,844303	0,222500	0,001728	35	1,302452	0,666536	0,004720
16	0,867210	0,166653	0,002486	36	1,325359	0,732087	0,004086
17	0,890118	0,117121	0,002614	37	1,348267	0,793651	0,004181
18	0,913025	0,075562	0,002742	38	1,371174	0,849233	0,004280
19	0,935933	0,042837	0,002870	39	1,394082	0,897684	0,004381
20	0,958840	0,019627	0,003000	40	1,416989	0,938006	0,004488
				41	1,439897	0,969366	0,004597

As can be seen from Table 1, the estimate holds:

$$\max_{j=\overline{1,41}} \| x_1(\hat{t}_j) - y(\hat{t}_j) \| < 0.0046.$$

Due to the fact that the algorithms of the Dzhumabaev parameterization method are convenient for numerical implementation, in the future they can become one of universal tools for identifying and finding approximate solutions to nonlinear boundary value problems.

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С.М. Темешева

Жұмабаевтың параметрлеу әдісі алгоритмдерінің модификациясы және сандық әдіс

Аңдатпа: Мақалада Д.С. Жұмабаевтың параметрлеу әдісі алгоритмдерінің бір модификациясы қарастырылады. Қосымша параметрлер кесіндінің ішкі бөліну нүктелерінде және кесіндінің екі ұшында да енгізіледі. Бұл алгоритмдердің жинақталуының жеткілікті шарттары бастапқы берілімдер терминдерінде келтіріледі. Дифференциалдық теңдеулер жүйесінің оң жағын және шекаралық шарт функциясын қолдана отырып, белгісіз параметрлердің бастапқы жуықтауын табу үшін сызықтық емес операторлық теңдеуі құрылады. Жай дифференциалдық теңдеулер жүйесі үшін сызықтық емес екі нүктелі шеттік есептің шешімін табудың сандық әдісі ұсынылады. Сандық әдіс тестілік мысалға қолданылады.

Түйінді сөздер: сызықтық емес екі нүктелі шеттік есеп, Жұмабаевтың параметрлеу әдісі, жеткілікті шарттар, оқшауланған шешім, сандық әдіс

С.М. Темешева

Модификация алгоритмов метода параметризации Джумабаева и численный метод

Аннотация. В статье рассматривается одна модификация алгоритмов метода параметризации Д.С. Джумабаева. Дополнительные параметры вводятся во внутренних точках разбиения отрезка и на обоих концах отрезка. Приведены достаточные условия сходимости этих алгоритмов в терминах исходных данных. С помощью правой части системы дифференциальных уравнений и функции краевого условия построено нелинейное операторное уравнение для нахождения начальных приближений неизвестных параметров. Предлагается численный метод нахождения решения нелинейной двухточечной краевой задачи для

системы обыкновенных дифференциальных уравнений. Численный метод реализован на тестовом примере.

Ключевые слова: нелинейная двухточечная краевая задача, метод параметризации Джумабаева, достаточные условия, изолированное решение, численный метод.

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SINGULAR BOUNDARY VALUE PROBLEMS FOR A NONLINEAR DIFFERENTIAL EQUATION

Abstract. *The paper deals with a nonlinear ordinary differential equation with singularities at the end-points of a finite interval. The definition of a limit with a weight solution is introduced and its attracting property is established. A singular boundary value problem for the differential equation is studied, where the boundary condition imposed on a solution is the requirement of its belonging to a functional ball centered at the limit solution.*

Key words: *nonlinear differential equation, singular boundary value problem, limit with a weight solution, approximation.*

On $(0,T)$, we consider a differential equation

$$\frac{dx}{dt} = f(t,x), \quad x \in \mathbb{R}^n, \quad \|x\| = \max_{i=\overline{1,n}} |x_i|, \tag{1}$$

where $f(t,x): (0,T) \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function with singularities at the endpoints mentioned in what follows in condition C_2.

Equations with singularities at the endpoint are often encountered in applications. Various problems for such equations have been studied by numerous authors (see [1-3] and references therein). To investigate the behavior of solutions of (1) at singular points, one can use so-called “limit solutions”.

In [4], for a nonlinear differential equation considered on the whole real line, the concept of a “limit solution as $t \to \infty$” was introduced. The conditions were established under which all solutions to the differential equation that belong to a functional ball coincide with a limit solution as $t \to \infty$. Using Lyapunov transformations and limit solutions, regular two-point boundary value problems were constructed that allow us, to a given degree of accuracy, to determine the restrictions of solutions bounded on the whole real line to a finite interval. To this end, iterative processes for unbounded operator equations [6] and the results obtained in [7] were used where analogous problems were studied for a linear ordinary differential equation.

It was proved that, under certain assumptions about the right-hand side of the equation, the limit solution $x_0(t)$ possesses an attracting property; i.e. there exists a functional ball centered at $x_0(t)$ where the differential equation has at least one solution, and all solutions from this ball coincide