системы обыкновенных дифференциальных уравнений. Численный метод реализован на тестовом примере.

Ключевые слова: нелинейная двухточечная краевая задача, метод параметризации Джумабаева, достаточные условия, изолированное решение, численный метод.

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SINGULAR BOUNDARY VALUE PROBLEMS FOR A NONLINEAR DIFFERENTIAL EQUATION

Abstract. *The paper deals with a nonlinear ordinary differential equation with singularities at the endpoints of a finite interval. The definition of a limit with a weight solution is introduced and its attracting property is established. A singular boundary value problem for the differential equation is studied, where the boundary condition imposed on a solution is the requirement of its belonging to a functional ball centered at the limit solution.*

Key words: *nonlinear differential equation, singular boundary value problem, limit with a weight solution, approximation.*

On $(0,T)$, we consider a differential equation

$$\frac{dx}{dt} = f(t,x), \quad x \in \mathbb{R}^n, \quad \|x\| = \max_{i=\overline{1,n}} |x_i|, \tag{1}$$

where $f(t,x): (0,T) \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function with singularities at the endpoints mentioned in what follows in condition C_2.

Equations with singularities at the endpoint are often encountered in applications. Various problems for such equations have been studied by numerous authors (see [1-3] and references therein). To investigate the behavior of solutions of (1) at singular points, one can use so-called "limit solutions".

In [4], for a nonlinear differential equation considered on the whole real line, the concept of a "limit solution as $t \to \infty$" was introduced. The conditions were established under which all solutions to the differential equation that belong to a functional ball coincide with a limit solution as $t \to \infty$. Using Lyapunov transformations and limit solutions, regular two-point boundary value problems were constructed that allow us, to a given degree of accuracy, to determine the restrictions of solutions bounded on the whole real line to a finite interval. To this end, iterative processes for unbounded operator equations [6] and the results obtained in [7] were used where analogous problems were studied for a linear ordinary differential equation.

It was proved that, under certain assumptions about the right-hand side of the equation, the limit solution $x_0(t)$ possesses an attracting property; i.e. there exists a functional ball centered at $x_0(t)$ where the differential equation has at least one solution, and all solutions from this ball coincide with $x_0(t)$. This property made it possible to solve the problem of approximation of a bounded on the whole real line solution to the differential equation.

Note that the attracting property of the limit solution $x_0(t)$ as $t \to \infty$ was established under the assumption that the differential equation linearized along the limit solution

$$\frac{dy}{dt} = f_x'(t, x_0(t))y, \quad y \in \mathbb{R}^n,$$

admits an exponential dichotomy on $\mathbb{R}$. However, in the case where the differential equation has certain singularities on its domain, it is necessary to take into account these features.

The concept of a limit solution was extended in [7] to the case of a nonlinear differential equation with a singularity at the left endpoint of the domain interval. In this case the limit solution was introduced with a weight that accounts for singularities of the equation considered. An attracting property of the limit with a weight solution was established.

This paper deals with singular boundary value problems for Eq. (1) on a finite interval. We define the concept of a limit solution at singular points and establish conditions under which this solution possesses an attracting property. We then construct approximating regular two-point boundary value problems that allow us to yield solutions of the singular boundary value problem of any specified accuracy.

Let *r* be a positive constant and $x_0(t)$ be a function continuously differentiable on $(0, T)$.

We will use the following notation:

$\breve{C}(\mathcal{J}, \mathbb{R}^n)$ is a space of functions $x: \mathcal{J} \to \mathbb{R}^n$ continuous and bounded on $\mathcal{J} \subseteq (0, T)$ with the norm $||x||_1 = sup_{t \in \mathcal{J}} ||x(t)||$;

$$S(x_0(t), \mathcal{J}, r) = \{x(t) \in \breve{C}(\mathcal{J}, \mathbb{R}^n): \ x(t) - x_0(t) \in \breve{C}(\mathcal{J}, \mathbb{R}^n), \|x - x_0\|_1 < r\};$$

$$G(x_0(t), \mathcal{J}, r) = \{(t, x): \ t \in \mathcal{J}, \|x - x_0\| < r\}.$$

The following conditions are assumed to be met:

C$_1$. The function $f_x'(t, x)$ is uniformly continuous in $G(x_0(t), (0, T), r)$.

C$_2$. The function $\alpha(t) = \|f_x'(t, x_0(t))\|$ satisfies the relations

$$\lim_{\delta \to 0+0} \int_{\delta}^{T/2} \alpha(t) dt = \infty, \quad \lim_{\delta \to 0+0} \int_{T/2}^{T-\delta} \alpha(t) dt = \infty;$$

C$_3$. $\lim_{t \to 0+0} \frac{f_x'(t, x_0(t))}{\alpha(t)} = A_0$, $\lim_{t \to T-0} \frac{f_x'(t, x_0(t))}{\alpha(t)} = A_T$, where A_0 and A_T are constant matrices whose eigenvalues have nonzero real parts: $\mathrm{Re}\ \lambda_i(A_0) \neq 0, \mathrm{Re}\ \lambda_i(A_T) \neq 0,\ i = \overline{1, n}$.

C$_4$. $\lim_{t \to 0+0} \frac{f(t,x)}{\alpha(t)} = f_0(x)$ and $\lim_{t \to T-0} \frac{f(t,x)}{\alpha(t)} = f_T(x)$.

Definition. *A function $x_T(t)$ continuously differentiable on $(0, T)$ is called a limit solution of Eq. (1) with weight $1/\alpha(t)$ as $t \to T - 0$ if*

$$\lim_{t \to T-0} \frac{\|\dot{x}_T(t) - f(t, x_T(t))\|}{\alpha(t)} = 0.$$

By S_0 and S_T we denote some real nonsingular ($n \times n$) matrices that reduce the matrices A_0 and A_T, respectively, to the generalized Jordan form

$$\tilde{A}_0 = S_0 A_0 S_0^{-1} = \begin{pmatrix} A_{11}^0 & 0 \\ 0 & A_{22}^0 \end{pmatrix}, \quad \tilde{A}_T = S_T A_T S_T^{-1} = \begin{pmatrix} A_{11}^T & 0 \\ 0 & A_{22}^T \end{pmatrix},$$

where A_{ii}^0 and A_{ii}^T, $i = 1,2$, consist of generalized Jordan boxes corresponding to the eigenvalues of the matrices A_0 and A_T with negative and positive real parts, respectively. Let n_1 and n_2 denote the respective numbers of eigenvalues of A_0 with negative real parts and eigenvalues of A_T with positive real parts.

Consider the problem

$$\frac{dx}{dt} = f(t,x), \qquad t \in [T-\delta, T), \ 0 < \delta < T, \tag{2}$$

$$P_T S_T[x(T-\delta) - x_T(T-\delta)] = d, \quad d \in \mathbb{R}^{n_2}, \tag{3}$$

$$x(t) \in S(x_T(t), [T-\delta, T), r_0), \quad r_0 > 0, \tag{4}$$

where $P_T = (0, I_{n_2})$ is an $(n_2 \times n)$ matrix.

Theorem 1. *Suppose that $x_T(t)$ is a limit solution with weight $1/\alpha(t)$ as $t \to T-0$ of Eq. (1) and the conditions C_1 –C_3 are satisfied. Then there exist numbers $\delta_0 \in (0,T)$, $r_0 > 0$, and $\rho_0 > 0$ such that, for any $\delta \in (0, \delta_0]$, problems (2) - (4) possesseses a unique solution for all $d \in \mathbb{R}^{n_2}$ satisfying the inequality $\|d\| < \rho_0$.*

The following theorem establishes an attracting property of the limit solution.

Theorem 2. *Suppose $x_T(t)$ is a limit solution of Eq. (1) with weight $1/\alpha(t)$ as $t \to T-0$ and conditions C_1 – C_3 are satisfied. Then:*

(i) there exist numbers $\delta_0 > 0$ and $r_0 \in (0,r]$ such that Eq. (1) has at least one solution in

$$S(x_T(t), [T-\delta_0, T), r_0);$$

(ii) any solution $x(t)$ of Eq. (1) belonging to $S(x_T(t), [T-\delta_0, T), r_0)$ satisfies the limit relation

$$\lim_{t \to T-0} \|x(t) - x_T(t)\| = 0.$$

In an analogous way, we can define a limit solution $x_0(t)$ of Eq. (1) with weight $1/\alpha(t)$ as $t \to 0+0$ and prove its attracting property.

We now proceed to a singular boundary value problem for Eq. (1). As a boundary condition we place the requirement on solutions to belong to a functional ball centered at a limit solution.

Problem 1 is to find a solution $\tilde{x}(t)$ to Eq. (1) that belongs to the functional ball

$$\tilde{x}(t) \in S\big(x_{0,T}(t), (0,T), r\big),$$

where $x_{0,T}(t)$ is a limit solution of Eq. (1) with weight $1/\alpha(t)$ as $t \to 0+0$ and $t \to T-0$.

To find an approximate solution to Problem 1 we pose the following problem.

Problem 2. Given $\varepsilon > 0$ it is required to determine a number $\delta > 0$ and a continuous function $g: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ for which a solution $x_\delta(t)$ of the two-point boundary value problem

$$\frac{dx}{dt} = f(t,x), \qquad t \in (\delta, T-\delta), \qquad x \in \mathbb{R}^n,$$

$$g[x(\delta), x(T-\delta)] = 0,$$

satisfies the inequality

$$\max_{t \in [\delta, T-\delta]} \|x_\delta(t) - x_{0,T}(t)\| < \varepsilon,$$

where $x_{0,T}(t)$ is a solution to Problem 1.

Let us construct the $(n \times n)$ matrices

$$P_1 = \begin{pmatrix} I_{n_1} & 0 \\ 0 & 0 \end{pmatrix} \text{ and } P_2 = \begin{pmatrix} 0 & 0 \\ 0 & I_{n_2} \end{pmatrix},$$

here I_{n_r} are the identity matrices of order n_r, $r = 1,2$. Under conditions C1-C4, we have constructed a regular two-point boundary value problem approximating Problem 1:

$$\frac{dx}{dt} = f(t,x), \qquad t \in (\delta, T-\delta), \qquad x \in \mathbb{R}^n,$$

$$P_1 S_0 f_0\big(x(\delta)\big) + P_2 S_T f_T\big(x(T-\delta)\big) = 0.$$

It can be shown that there is a mutual relationship between the solvability of Problem 1 and that of its approximating two-point regular boundary value problem.

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Утешова Р.Е.

Бейсызықты дифференциалдық теңдеу үшін сингулярлы шеттік есептер

Аңдатпа: Оң жағының аралықтың шеттерінде елеулі ерекшеліктері бар бей сызықты жай дифференциалдық теңдеу қарастырылады. Салмақты шегі бар шешімнің ұғымы енгізілген, оның тарту қасиеті анықталды. Берілген теңдеу үшін сингулярлы шеттік есеп зерттелінеді, оның шекаралық шарты шешімінің белгілі бір функционалды шарға жататындығы болып табылады.

Түйінді сөздер: бейсызықты дифференциалдық теңдеу, сингулярлы шеттік есеп, салмақты шегі бар шешім, аппроксимация

Утешова Р.Е.

Сингулярные краевые задачи для нелинейного дифференциального уравнения

Аннотация. В статье рассматривается нелинейное обыкновенное дифференциальное уравнение с особенностями в конечных точках интервала. Вводится понятие предельного с весом решения и устанавливается его притягивающее свойство. Изучается сингулярная краевая задача для данного уравнения, в которой качестве краевого условия выступает требова-

ние принадлежности решения к некоторому функциональному шару, центром которого является предельное с весом решение.

Ключевые слова: нелинейное дифференциальное уравнение, сингулярная краевая задача, предельное с весом решение, аппроксимация.

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NUMERICAL SOLUTION OF THE BOUNDARY VALUE PROBLEMS FOR THE LOADED DIFFERENTIAL AND FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS

Abstract. *The article presents a computational method to solve boundary value problems for the loaded differential and Fredholm integro-differential equations. Solving a problem for the loaded differential and Fredholm integro-differential equations is reduced to solving a system of linear algebraic equations in relation to the additional parameters introduced. A numerical method for finding a solution of the problem is suggested, which is based on solving the constructed system and the Bulirsch-Stoer method for solving Cauchy problems on the subintervals. The result is illustrated by example.*

Key words: *integro-differential equation, loaded differential equation, parametrization method, numerical method.*

Introduction. Loaded differential equations are used to solve problems of long-term prediction and control of the groundwater level and soil moisture [1, 2]. Various problems for loaded differential equations and methods of finding their solutions are considered in [1, 3-8].

A new concept of a general solution of a linear loaded differential equation was proposed in [9]. A new general solution was introduced for the linear Fredholm integro-differential equation in [10]. Replacing the integral term of an integro-differential equation with a quadrature formula also leads to a loaded differential equation. Therefore, numerical and approximate methods for solving boundary value problems for loaded differential equations are also used in solving boundary value problems for integro-differential equations.

On the basis of the parametrization method [11], in [10], a new approach to the general solution of the linear Fredholm integro-differential equation was proposed. The interval where the equation is considered is divided into parts, and the values of the solution at the starting points of subintervals are taken as additional parameters. With the help of newly introduced unknown functions on the sub-intervals, a special Cauchy problem for a system of integro-differential equations with parameters is received. Using the solution of the special Cauchy problem, a new general solution of