УДК 517.929.7.

Narkesh B. Iskakova[1,*], Nurgul T. Orumbayeva[2], Nazira Nurzhuma[1]

[1] Abai Kazakh National Pedagogical University, Almaty, Kazakhstan
[2] E.A.Buketov University of Karaganda, Karaganda, Kazakhstan

A PERIODIC BOUNDARY VALUE PROBLEM FOR A SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS WITH A DELAY ARGUMENT

Abstract. *In the paper, a periodic boundary value problem for a system of linear differential equations with a delay argument is considered. On the basis of the parameterization method, a two-parametric family of algorithms for finding a solution of the periodic boundary value problem is offered.*

Key words: *parameterization method, differential equations, delay argument, algorithm, unique solution*

In various applications, there has been an increasing interest in the theory of linear boundary value problems for differential equations with a delay argument. Due to applications in physics, biology, epidemiology, and other fields, most of the literature on delay differential equations has been focused on the existence of a periodic solution, oscillations, etc. An analysis of the literature indicates that in recent decades, boundary value problems for delay differential equations have been extensively studied; see, for example, [1-4].

We consider the periodic boundary value problem for the system of linear differential equations with a delay argument

$$\frac{dx(t)}{dt} = A(t)x(t) + B(t)x(t-\tau) + f(t), \quad t \in [0,T], \quad x \in R^n, \tag{1}$$

$$x(z) = diag[x(0)] \cdot \varphi(z), \quad z \in [-\tau, 0], \tag{2}$$

$$x(0) = x(T), \tag{3}$$

where the matrices $A(t), B(t)$, and the vector function $f(t)$ are continuous on $[0,T]$, $\varphi(t)$ is a continuous vector function given on the initial set $[-\tau,0]$, such that $\varphi_i(0) = 1,\ i = \overline{1,n}$, $\tau > 0$ is a constant delay, $\|A(t)\| = \max\limits_{i=\overline{1,n}} \sum\limits_{j=1}^{n} \|a_{ij}(t)\| \le \alpha$, $\|B(t)\| \le \beta$, $\alpha, \beta - const.$, $\|x(t)\| = \max\limits_{i=\overline{1,n}} |x_i|$.

A solution of problems (1)-(3) is a vector-function $x(t)$ continuous on $[-\tau,T]$ and continuously differentiable on $[-\tau,0) \cup (0,T]$ that satisfies the system of differential equations (1) on $[0,T]$, coincides with the function $diag[x(0)] \cdot \varphi(t)$ on $[-\tau,0]$, and have the values at the points $t=0, t=T$ for which equality (3) is valid.

Using the parameterization method [5], a partition of the interval $[-\tau,T)$ is performed with step size $h = \tau/l$: $N\tau = T, l \in \mathrm{N}$

$$[-\tau,0) \cup [0,T) = \bigcup_{s=l}^{1} [-t_s, -t_{s-1}) \cup \bigcup_{r=1}^{lN} [t_{r-1}, t_r),$$

where $t_0 = 0$, $-t_s = -sh$, $s = \overline{1,l}$, $t_r = rh$, $r = \overline{1,lN}$.

The restriction of the function $x(t)$ to the r-th subinterval $[t_{r-1}, t_r)$ is denoted by $x_r(t)$, $r = \overline{1,lN}$. By $\varphi_s(t)$, $s = \overline{1,l}$, we denote the restrictions of the initial function $\varphi(t)$ to the s-th subinterval $[-t_{l-(s-1)}, -t_{l-s})$. If the argument $t-\tau$ is changed on $[t_{r-1-l}, t_{r-l})$, then

$$[t_{r-1-l}, t_{r-l}) = \begin{cases} [t_{r-1-l}, t_{r-l}), & \text{if } r-1-l \geq 0, r-l \geq 1, \\ [-t_{l-(r-1)}, -t_{l-r}), & \text{if } r-1-l < 0, r-l < 1, \end{cases}$$

therefore, the function $x(t-\tau)$ is the same as the function $x_{r-l}(t-\tau)$.

Introducing the parameters $\lambda_r = x(t_{r-1})$ and making the substitution the functions $u_r(t) = x(t) - \lambda_r$ on each r-th subinterval, we get the boundary-value problem with parameters:

$$\frac{du_r}{dt} = A(t)(u_r(t)+\lambda_r) + B(t)\Phi_r(t-\tau)\lambda_1 + f(t), \quad t \in [t_{r-1}, t_r),\ r = \overline{1,l}, \tag{4}$$

$$\frac{du_r}{dt} = A(t)(u_r(t)+\lambda_r) + B(t)(u_{r-l}(t-\tau)+\lambda_{r-l}) + f(t), \quad r = \overline{l+1, lN}, \tag{5}$$

$$u_r(t_{r-1}) = 0, \ r = \overline{1, lN}, \tag{6}$$

$$\lambda_1 = \lambda_{lN} + \lim_{t \to T-0} u_{lN}(t), \tag{7}$$

$$\lambda_s + \lim_{t \to t_s - 0} u_s(t) = \lambda_{s+1}, \quad s = \overline{1, lN-1}, \tag{8}$$

where $\Phi_r(t-\tau)$ is an $(n \times n)$ matrix of the form $diag[\varphi_r(t-\tau)]$, $r = \overline{1,l}$.

If $x(t)$ is a solution of problem (1)-(3), then the system of pairs $(\lambda_r, u_r(t))$, $r = \overline{1, lN}$, is a solution to problems (4)-(8). Conversely, if the system $(\tilde{\lambda}_r, \tilde{u}_r(t))$, $r = \overline{1, lN}$, is a solution of problems (4)-(8), then the function

$$\tilde{x}(t) = \begin{cases} \tilde{\lambda}_r + \tilde{u}_r(t), \ t \in [t_{r-1}, t_r), \ r = \overline{1, lN}, \\ \tilde{\lambda}_{lN} + \lim_{t \to T-0} \tilde{u}_{lN}(t), \ t = T, \end{cases}$$

is a solution of problem (1)-(3).

In problems (4)-(8), initial conditions (6) appeared that allow us to determine unknown functions from the Volterra integral equations of the 2-nd kind. The functions $u_r(t)$, $t \in [t_{r-1}, t_r)$, $r = \overline{1,l}$, for a fixed parameter λ_r, are defined from the equation

$$u_r(t) = \int_{t_{r-1}}^{t} A(s)[u_r(s)+\lambda_r]ds + \int_{t_{r-1}}^{t} B(s)\Phi_r(s-\tau)ds + \int_{t_{r-1}}^{t} f(s)ds, \tag{9}$$

and the function $u_r(t)$, $t \in [t_{r-1}, t_r)$, $r = \overline{l+1, lN}$, for a fixed $\lambda_r, \lambda_{r-l}, u_{r-l}(t-\tau)$, is defined from the equation

$$u_r(t) = \int_{t_{r-1}}^{t} A(s)[u_r(s)+\lambda_r]ds + \int_{t_{r-1}}^{t} B(s)[u_{r-l}(s-\tau)+\lambda_{r-l}]ds + \int_{t_{r-1}}^{t} f(s)ds, \tag{10}$$

where the pair $(\lambda_r, u_r(t))$, $r = \overline{1,l}$, satisfies equation (9), and the pairs $(\lambda_{r-l}, u_{r-l}(t))$, $r = l+1, l+2, ..., l(N-1)$, satisfy the equations

$$u_{r-l}(t) = \int_{t_{r-l-1}}^{t} A(s)[u_{r-l}(s)+\lambda_{r-l}]ds + \int_{t_{r-l-1}}^{t} B(s)[u_{r-2l}(s-\tau)+\lambda_{r-2l}]ds + \int_{t_{r-l-1}}^{t} f(s)ds, \qquad t \in [t_{r-l-1}, t_{r-l}).$$

In (9), replacing $u_r(s)$ by the right-hand side of this equation, and repeating the process ν $(\nu = 1,2,...)$ times, we get the following representation of the function $u_r(t)$, $t \in [t_{r-1}, t_r)$, $r = \overline{1,l}$:

$$u_r(t) = D_{\nu r}(t,0)\lambda_r + E_{\nu r}(t,0)\lambda_1 + F_{\nu r}(t, f_0) + G_{\nu r}(t, u_{r,0}). \tag{11}$$

In the same way, from (10) we get the following representation of $u_{il+j}(t)$, $i=\overline{1,N-1}$, $j=\overline{1,l}$, :

$$u_{il+j}(t)=D_{\nu,il+j}(t,0)\lambda_{il+j}+P^{i}_{\nu,il+j}[t,E_{\nu,il+j}(t,i\tau)]\cdot\lambda_1+$$
$$+\sum_{k=1}^{i}P^{k-1}_{\nu,il+j}[t,H_{\nu,il+j}(t,(k-1)\tau)+P_{\nu,il+j}[t,D_{\nu,il+j}(t,k\tau)]]\cdot\lambda_{(i-k)l+j}+ \quad (12)$$
$$+\sum_{k=0}^{i}P^{k-1}_{\nu,il+j}[t,F_{\nu,il+j}(t,f_{k\tau})+G_{\nu,il+j}(t,u_{(i-k)l+j,k\tau})],\ t\in[t_{il+j-1},t_{il+j}),$$

where

$$D_{\nu,il+j}(t,m\tau)=\sum_{k=0}^{\nu-1}\int_{t_{il+j-1}}^{t}A(s_1-m\tau)\ldots\int_{t_{il+j-1}}^{s_k}A(s_{k+1}-m\tau)ds_{k+1}..ds_1,$$

$$H_{\nu,il+j}(t,m\tau)=\int_{t_{il+j-1}}^{t}B(s_1-m\tau)ds_1+\sum_{k=1}^{\nu-1}\int_{t_{il+j-1}}^{t}A(s_1-m\tau)\ldots$$
$$\ldots\int_{t_{il+j-1}}^{s_{k-1}}A(s_k-m\tau)\int_{t_{il+j-1}}^{s_k}B(s_{k+1}-m\tau)ds_{k+1}\,ds_k\ldots ds_1$$

$$F_{\nu,il+j}(t,f_{m\tau})=\int_{t_{il+j-1}}^{t}f(s_1-m\tau)ds_1+\sum_{k=1}^{\nu-1}\int_{t_{il+j-1}}^{t}A(s_1-m\tau)\ldots$$
$$\ldots\int_{t_{il+j-1}}^{s_{k-1}}A(s_k-m\tau)\int_{t_{il+j-1}}^{s_k}f(s_{k+1}-m\tau)ds_{k+1}\,ds_k\ldots ds_1,$$

$$G_{\nu,il+j}(t,u_{il+j,m\tau})=\int_{t_{il+j-1}}^{t}A(s_1-m\tau)\ldots\int_{t_{il+j-1}}^{s_{\nu-2}}A(s_{\nu-1}-m\tau)\int_{t_{il+j-1}}^{s_{\nu-1}}A(s_{\nu-1}-m\tau)u_{il+j}(s_\nu)ds_\nu ds_{\nu-1}..ds_1,$$

$$P_{\nu,il+j}(t,u_{(i-1)l+j,m\tau})=\int_{t_{il+j-1}}^{t}B(s_1-(m-1)\tau)u_{(i-1)l+j}(s_1-m\tau)ds_1+\sum_{k=1}^{\nu-1}\int_{t_{il+j-1}}^{t}A(s_1-(m-1)\tau)\ \ldots$$
$$\ldots\int_{t_{il+j-1}}^{s_{k-1}}A(s_k-(m-1)\tau)\int_{t_{il+j-1}}^{s_k}B(s_{k+1}-(m-1)\tau)u_{(i-1)l+j}(s_{k+1}-m\tau)ds_{k+1}\,ds_k\ldots ds_1,$$

$$E_{\nu,il+j}(t,m\tau)=\int_{t_{il+j-1}}^{t}B(s_1-m\tau)\Phi_j(s_1-(m+1)\tau)ds_1+\sum_{k=1}^{\nu-1}\int_{t_{il+j-1}}^{t}A(s_1-m\tau)\ \ldots$$
$$\ldots\int_{t_{il+j-1}}^{s_{k-1}}A(s_k-m\tau)\int_{t_{il+j-1}}^{s_k}B(s_{k+1}-m\tau)\Phi_j(s_{k+1}-(m+1)\tau)ds_{k+1}\,ds_k\ldots ds_1,$$

$$m=\overline{0,i},\ i=\overline{1,N-1},\ j=\overline{1,l},\ P^0[t,y]=y,\ P^k[t,y]=P[t,P^{k-1}[t,y]].$$

In (11) and (12), passing to the limits and substituting them in the boundary conditions (7) and the continuity conditions (9), we obtain a system of linear algebraic equations in unknown parameters $\lambda_1,\lambda_2,\ldots,\lambda_{lN}$. This system can be written in the matrix form

$$Q_\nu(l)\cdot\lambda=-\widetilde{F}_\nu(f,l)-\widetilde{G}_\nu(u,l), \quad (13)$$

here $Q_\nu(l)$ is an ($nlN \times nlN$) matrix composed of the coefficients of unknown parameters

$$\lambda_r,\ r=\overline{1,lN},\quad \lambda=\left(\lambda_1,\ \lambda_2,\ldots,\lambda_{lN}\right)'\in R^{nlN},$$

$$\tilde{F}_\nu(l)=\left(-\tilde{F}_{\nu,lN}(T),\tilde{F}_{\nu 1}(t_1),\tilde{F}_{\nu 2}(t_2),\ldots,\tilde{F}_{\nu,lN-1}(t_{lN-1})\right)'\in R^{nlN},$$

$$\tilde{G}_\nu(u,l)=\left(-\tilde{G}_{\nu,lN}(u,T),\tilde{G}_{\nu 1}(u,t_1),\tilde{G}_{\nu 2}(u,t_2),\ldots,\tilde{G}_{\nu,lN-1}(u,t_{lN-1})\right)'\in R^{nlN},$$

$$\tilde{F}_{\nu,il+j}(t_{il+j})=\sum_{k=0}^{i}P^k_{\nu,il+j}\left[t_{il+j},F_{\nu,il+j}\left(t,f_{k\tau}\right)\right],$$

$$\tilde{G}_{\nu,il+j}(u,t_{il+j})=\sum_{k=0}^{i}P^k_{\nu,il+j}\left[t_{il+j},G_{\nu,il+j}\left(t,u_{(i-k)l+j,k\tau}\right)\right],\ i=\overline{0,N-1},\ j=\overline{1,l}.$$

Thus, we have the system of equations (9), (10) and (13) for finding the pair $\left(\lambda,u[t]\right)$, where $\lambda=\left(\lambda_1,\ \lambda_2,\ldots,\lambda_{lN}\right)$, $u[t]=\left(u_1(t),\ u_2(t),\ldots,u_{lN}(t)\right)$.

We find a solution $\left(\lambda,u[t]\right)$ of problems (4)-(8) as the limit of the sequence $\left(\lambda^{(k)},u^{(k)}[t]\right)$, $k=0,1,2,\ldots$, using the following algorithm:

Step 0. (a) Assuming that the matrix $Q_\nu(l)$ is invertible for some ν and l, the initial approximation for the parameter $\lambda^{(0)}=\left(\lambda_1{}^{(0)},\ \lambda_2{}^{(0)},\ldots,\lambda_{lN}{}^{(0)}\right)$ is determined from the equation $Q_\nu(l)\cdot\lambda=-\tilde{F}_\nu(l)$, that is $\lambda^{(0)}=-[Q_\nu(l)]^{-1}\tilde{F}_\nu(l)$;

(b) By solving the Cauchy problems (4) and (6) on $[t_{r-1},t_r)$ with $\lambda_r=\lambda_r^{(0)}$, we find $u_r^{(0)}(t)$, $r=\overline{1,l}$. Substituting $\lambda_r,\lambda_{r-l},u_{r-l}(t-\tau)$ in (5) by the corresponding values $\lambda_r^{(0)},\lambda_{r-l}^{(0)},u_{r-l}^{(0)}(t-\tau)$ and solving the Cauchy problems (5) and (6) on $[t_{r-1},t_r)$, $r=\overline{l+1,lN}$, we find $u_r^{(0)}(t)$, $r=\overline{l+1,lN}$.

Step 1. (a) Substituting $u_r^{(0)}(t)$ found above in the right-hand side of (13), we determine $\lambda^{(1)}$ from the equation $Q_\nu(l)\cdot\lambda=-\tilde{F}_\nu(f,l)-\tilde{G}_\nu(u^{(0)},l)$;

(b) on the interval $[t_{r-1},t_r)$, solving the Cauchy problems (4) and (6) with $\lambda_r=\lambda_r^{(1)}$, we find $u_r^{(1)}(t)$, $r=\overline{1,l}$. Substituting $\lambda_r,\lambda_{r-l},u_{r-l}(t-\tau)$ by $\lambda_r^{(1)},\lambda_{r-l}^{(1)},u_{r-l}^{(1)}(t-\tau)$, respectively, we solve the Cauchy problems (5) and (6) on the interval $[t_{r-1},t_r)$, $r=\overline{l+1,lN}$, and find $u_r^{(1)}(t)$, $r=\overline{l+1,lN}$.

And so on. Continuing the process, in the k–th step, we get a system of pairs $\left(\lambda^{(k)},u^{(k)}[t]\right)$. Sufficient conditions for convergence and feasibility of the proposed algorithm is established.

Theorem 1. Let for some $l,\ l\in N$, and $\nu,\ \nu\in N$ the matrix $Q_\nu(l)$: $RnlN\to RnlN$ be invertible and the inequalities

(a) $$\left\|[Q_\nu(l)]^{-1}\right\|\le\gamma_\nu(l);$$

(b) $$q_\nu(l)=\gamma_\nu(l)\frac{1}{\nu!}\left(\frac{\alpha\tau}{l}\right)^{\nu}\max_{i=\overline{0,N-1}}\sum_{\rho=0}^{i}\frac{1}{\rho!}\left(\frac{\beta\tau}{l}\sum_{k_1=0}^{\nu-1}\frac{1}{k_1!}\left(\frac{\alpha\tau}{l}\right)^{k_1}\right)^{\rho}\cdot P(l)<1,$$

hold, where

$$P(l)=\max\left\{\max_{1\le j\le l}\sup_{t\in[t_{j-1},t_j}\left\{e^{\frac{\alpha\tau}{l}}-1+\frac{\beta\tau}{l}\cdot e^{\frac{\alpha\tau}{l}}\left\|\Phi_j(t-(i+1)l)\right\|\right\}\right\}.$$

$$\max_{\substack{i=\overline{0,N-1},\\ j=\overline{1,l}}} \sup_{t\in[t_{il+j-1},t_{il=j})} \left\{ e^{\frac{\alpha\tau}{l}} \sum_{k_1=0}^{i} \left(\frac{\beta\tau}{l} \cdot e^{\frac{\alpha\tau}{l}} \right)^{k_1} + e^{\frac{\alpha\tau}{l}} - 1 + \left(\frac{\beta\tau}{l} \cdot e^{\frac{\alpha\tau}{l}} \right)^{i+1} \left\| \Phi_j(t-(i+1)l) \right\| \right\}.$$

Then the sequence of pairs $\left(\lambda^{(k)}, u^{(k)}[t]\right)$ converges to a unique solution $\left(\lambda^*, u^*[t]\right)$ of the problem (4)-(8) as $k \to \infty$.

Due to the equivalence of problems (1)-(3) and (4)-(8), the following statement holds true.

Theorem 2. Let the conditions of Theorem 1 be fulfilled. Then problem (1)-(3) has a unique solution $x^*(t)$ and the estimate

$$\left\| x^*(t) - x^{(k)}(t) \right\| \le \gamma_\nu(l) \frac{(q_\nu(l))^k}{1-q_\nu(l)} \cdot \frac{1}{\nu!} \left(\frac{\alpha\tau}{l} \right)^\nu \times$$

$$\times \max_{i=\overline{0,N-1}} \sum_{\rho=0}^{i} \frac{1}{\rho!} \left(\frac{\beta\tau}{l} \sum_{k=0}^{\nu-1} \frac{1}{k!} \left(\frac{\alpha\tau}{l} \right)^k \right)^\rho \cdot M(l)(1+P(l)), \quad t \in [0,T],$$

is valid, where $x^{(k)}(t)$ is a function piecewise continuously differentiable on $[0,T]$, for which the function $\lambda_r^{(k)} + u_r^{(k)}(t)$, $r = \overline{1,lN}$, $k = 0,1,2,...$ is a restriction on $[t_{r-1}, t_r)$, $r = \overline{1,lN}$.

REFERENCES

1. ShuX.-B., Li-HongH., and LiY.-J., "Triple positive solutions for a class of boundary value problems for second-order neutral functional differential equations", *Nonlinear Analysis: Theory, Methods & Applications*, 65:4 (2006), 825-840.
2. LiuY., "Periodic boundary value problems for first order functional differential equations with impulse", *Journal of Computational and Applied Mathematics*, 223:1 (2009), 27-39.
3. Feng M. Q., "Periodic solutions for prescribed mean curvature Lienard equation with a deviating argument", *Nonlinear Analysis: Real World Applications*, 13:3 (2012), 1216-1223.
4. WangH., "Positive Solution for a Class of Boundary Value Problems with Finite Delay", *Journal of Applied Mathematics: Hindawi Publishing Corporation*, Vol. 2012, Article ID 382392, 7 pages.
5. Dzhumabaev D.S., "Conditions the unique solvability of a linear boundary value problem for a ordinary differential equations", *Computational Mathematics and Mathematical Physics*, 29:1 (1989), 34-46.

Dedicated to the bright memory of an outstanding scientist,
Doctor of Physical and Mathematical Sciences, Professor,
our scientific supervisor Dzhumabaev Dulat Syzdykbekovich

Н.Б. Искакова, Н.Т. Орумбаева, Н. Нұржума

Кешігулі аргументі бар сызықтық дифференциалдық теңдеулер жүйесі үшін периодтық шеттік есеп

Аңдатпа: Мақалада кешігулі аргументі бар сызықтық дифференциалдық теңдеулер жүйесі үшін периодтық шеттік есеп қарастырылады. Параметризация әдісі негізінде периодтық шеттік есептің шешімін табу үшін алгоритмдердің екі параметрлік тобы ұсынылады.

Түйінді сөздер: параметризация әдісі, дифференциалдық теңдеулер, кешігулі аргумент, алгоритм, жалғыз шешім

Н.Б. Искакова, Н.Т. Орумбаева, Н. Нұржума

Периодическая краевая задача для системы линейных дифференциальных уравнений с запаздывающим аргументом

Аннотация. В статье рассматривается периодическая краевая задача для системы линейных дифференциальных уравнений с запаздывающим аргументом. На основе метода параметризации предлагается двухпараметрическое семейство алгоритмов для нахождения решения периодической краевой задачи.

Ключевые слова: метод параметризации, дифференциальные уравнения, запаздывающий аргумент, алгоритм, единственное решение.

Сведения об авторах:

Искакова Наркеш Билаловна, к.ф.-м.н., и.о. ассоциированного профессора кафедры математики и математического моделирования Казахского Национального педагогического университета имени Абая.

Орумбаева Нургуль Тумарбековна, к.ф.-м.н., доцент кафедры математического анализа и дифференциальных уравнений Карагандинского университета имени Е.А. Букетова.

Нұржума Назира, магистрант 2-го курса Казахского Национального педагогического университета имени Абая, специальность 7М05401-Математика.

About the authors:

Narkesh B. Iskakova, candidate of physical and mathematical Sciences, acting associate Professor, Department of mathematics and mathematical modeling, Abay Kazakh National pedagogical University.

Nurgul T. Orumbayeva, candidate of physical and mathematical Sciences, associate Professor, Department "Mathematical analysis and differential equations", E.A. Buketov University of Karaganda.

Nazira Nurzhuma, master's student of the 2nd year of Abay Kazakh National pedagogical University, specialty 7M05401-Mathematics.