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ON THE CORRECT SOLVABILITY OF A LINEAR TWO-POINT BOUNDARY VALUE PROBLEM WITH IMPULSE ACTION BY THE PARAMETERIZATION METHOD

Abstract. *A linear two-point boundary value problem with impulse action for a system of ordinary differential equations is considered. The necessary and sufficient conditions for the correct solvability of the problem are established.*

Key words: *boundary value problem, impulse, parameterization method, correct solvability*

Mathematical modeling of the evolution of real processes with short-term disturbances, the duration of which can be neglected, leads to the need to study differential equations with impulse effects.

Such problems attracted the attention of scientists as early as the end of the 19th and beginning of the 20th centuries, at the stage of the formation of nonlinear mechanics, and first of all aroused the attention of physicists with the possibility of an adequate description of processes in nonlinear oscillatory systems. Intensive development of the latest technology has led to an increase in the interest of mathematicians in the further study of systems with discontinuous trajectories. Examples of applications include pulse control systems, pulse computing systems, etc. The research results are used in many engineering, technical, economic, biomedical and other problems.

It is known that the presence of an impulse significantly affects the solvability properties of boundary value problems and the properties of their solutions. For example, Perestyuk showed the positive effect of the impulse action on the continuability of a solution of a nonlinear ordinary differential equation. The dissertation contains examples showing both the positive and negative influence of the impulse on the solvability of periodic boundary-value problem for an ordinary differential equation. This fact, among others, generates the urgent problem of researching properties and developing algorithms for finding solutions to boundary value problems with impulse effect. The qualitative theory of differential equations with impulse effects dates back to works by A. D. Myshkis, A. M. Samoilenko [3], A. Khalanai, D. Veksler [4] and other mathematicians. The theory of boundary value problems for ordinary differential equations with impulse action was substantially developed in the works of the Kiev school of mathematicians. Boundary and periodic boundary value problems for ordinary differential equations with impulse effect are considered in the works of A.M. Samoilenko, A.A. Perestyuk, Shavkoplyas, Trofimchuk, Rogovchenko, Karanjulov and other mathematicians. Various methods have been developed and applied by them both to investigate the solvability of boundary value problems with impulsive effect, and to find their solution.

In particular, a numerical - analytical method proposed by A.M.Samoilenko for ordinary differential equations with impulsive effect is widely used. We note that using the general solution of a system of ordinary differential equations allows us to obtain the necessary and sufficient conditions for the unique solvability of a linear boundary-value problem with impulse effect in terms of the fundamental matrix. However, given that it is possible to construct the fundamental matrix for systems of differential equations with variable coefficients in rare cases, this criterion for the unique solvability is applicable only for a narrow class of boundary value problems. For nonlinear boundary value problems with impulsive effect, only sufficient conditions for their solvability are established, which allow us to study classes of boundary value problems that satisfy certain assumptions. If the general solution of the considered system of nonlinear ordinary differential equations is known, then using the conditions of the impulse effect condition and boundary conditions, we can construct a system of nonlinear equations with respect to arbitrary constants. The solvability of the problem will be equivalent to the existence of a solution to the constructed system. Since for non-linear systems of ordinary differential equations, as a rule, a general solution cannot be found, this sign of solvability of a nonlinear boundary value problem with impulsive effect is applicable in exceptional cases.

Therefore, we study the following questions:

1) To obtain coefficient criteria for the unique solvability of a linear two-point boundary value problem with impulse action without using fundamental matrix;

2) To establish necessary and sufficient conditions for the existence of an isolated solution of a periodic boundary problem for a nonlinear system of ordinary differential equations with impulse action in terms of the input data;

3) To build effective algorithms for solving boundary value problems for systems of ordinary differential equations with impulse action.

These issues are resolved based on the parameterization method proposed by D.S. Dzhumabaev.

On the interval $[0,T]$, we consider the linear two-point boundary value problem with an impulse action at fixed moments of time for a system of ordinary differential equations

$$\frac{dx}{dt} = A(t)x + f(t),\ t \in [0,T]\backslash\{\theta_1,\theta_2,\dots,\theta_m\},\ \theta_i \in (0,T),\ i=\overline{1,m},\ x \in R^n \quad (1)$$

$$B_0x(0) + C_0x(T) = d,\ d \in R^n, \quad (2)$$

$$B_ix(\theta_i - 0) - C_ix(\theta_i + 0) = p_i,\ p_i \in R^n, \quad (3)$$

where the matrix $A(t)$ and the vector function $f(t)$ are piecewise continuous on $[0,T]$ with possible discontinuity points of the first kind θ_i, $i=\overline{1,m}$, B_i, C_i, $i=(\overline{0,m})$ are constant matrices; $\|x\| = \max_i |x_i|$, $\|A(t)\| = \max_i \sum_{j=1}^n |a_{ij}(t)| \le \alpha$, $\|f(t)\|_1 = \max_{t\in[0,T]} \|f(t)\|$.

A solution of problem (1)-(3) is piecewise continuously differentiable on $[0,T]$ vector function $x(t)$, which satisfies the differential equation (1) on $[0,T]$ except points θ_i, as well as conditions (2) and (3). By $\tilde{C}([0,T],R^n)$ we denote the space of piecewise continuous on $[0,T]$ function $x\colon [0,T] \to R^n$ with the norm $\|x\|_2 = \max_{i\in[0,m]} \sup_{t\in[\theta_i,\theta_{i+1})} \|x(t)\|$, where $\theta_0 = 0,\ \theta_{m+1} = T$.

The need to study boundary value problems for systems of differential equations with impulse action is caused by many problems of physics, engineering and biology, which describe real processes subject to impulse effect. A review and bibliography of works devoted to the study of systems of differential equations with impulsive effect can be found in [1-6]. In [7], a periodic problem with a pulsed effect at an internal point of the interval was studied by the parameterization method [8]. Algorithms for finding a solution are proposed and sufficient conditions for their convergence are established, that provide the unique solvability of the problem under consideration. In [9], algorithms were proposed for finding a solution to problems (1) - (3) in terms of the matrix $Q_\nu(h_1,h_2,\dots,h_{m+1})$, $\nu \in N$, $h_j = \theta_j - \theta_{j-1}$, $j=\overline{1,m+1}$, $\theta_0 = 0$, $\theta_{m+1} = T$, composed of the matrices $A(t)$, B_i, C_i, $i=(\overline{0,m})$, and θ_j, the conditions for their convergence were obtained. Criteria for the unique solvability of problem (1) - (3) were established in terms of the matrix $Q_\nu(h_1,h_2,\dots,h_{m+1})$. In the present work, the effect of changing the partition step for a fixed algorithm for convergence, ν, leads to the unique solvability of problems (1) - (3).

Let us take a number $l \in N$ and make the partition $[0,T) = \bigcup_{r=1}^{(m+1)l} [t_{r-1},t_r)$, $t_0 = 0$, $t_r = t_{r-1} + \frac{h_1}{l}$, $r=\overline{1,l}$, $t_r = t_{r-1} + \frac{h_2}{l}$, $r=\overline{l+1,2l}$, …, $t_r = t_{r-1} + \frac{h_{m+1}}{l}$, $r=\overline{ml+1,(m+1)l}$, $h^0 = \max_{i=\overline{1,m+1}} h_i$, $h_i = \min_{i=\overline{1,m+1}} h_i$, $\delta = \frac{h^0}{h_0}$.

By $x_r(t)$ we denote the restriction of the function $x(t)$ to the partition subinterval $[t_{r-1}, t_r)$ and reduce problems (1) - (3) to a multipoint boundary value problem with an impulse effect.

$$\frac{dx_r}{dt} = A(t)x_r + f(t), \quad t \in [t_{r-1}, t_r), \quad r = \overline{1,(m+1)l}, \tag{4}$$

$$B_0 \lim_{t\to t_{il}-0} x_1(0) + C_0 \lim_{t\to T-0} x_{(m+1)l}(t) = d, \tag{5}$$

$$B_i \lim_{t\to t_{il}-0} x_{il}(t) - C_i x_{il}(t_{il+1}) = p_i, \quad i = \overline{1,m}. \tag{6}$$

$$\lim_{t\to t_s-0} x_s(t) = x_{s+1}(t_s), \quad s = \{\overline{1,(m+1)l-1}\}\setminus\{il\}, \quad i = \overline{1,m}. \tag{7}$$

Here (7) are the conditions for matching the solution at the interior points of the partition.

If $x(t)$ is a solution to problems (1) - (3), then the system of its restrictions $x[t] = (x_1(t), x_2(t), \dots, x_{(m+1)l}(t))'$ is a solution to problems (4) - (7). And vice versa, if the system of vector functions $\tilde{x}[t] = (\tilde{x}_1(t), \tilde{x}_2(t), \dots, \tilde{x}_{(m+1)l}(t))'$ is a solution to problems (4)-(7), then the function $\tilde{x}(t)$, defined by the equalities $\tilde{x}(t) = \tilde{x}_r(t)$, $t \in [t_{r-1}, t_r)$, $r = \overline{1,(m+1)l}$, $\tilde{x}(T) = \lim_{t\to T-0} \tilde{x}_{(m+1)l}(t)$, will be the solution to the original problem. We introduce the notation $\lambda_r = x_r(t_{r-1})$ and make the substitution $u_r(t) = x_r(t) - \lambda_r$ at each subinterval $[t_{r-1}, t_r)$. Then the problems (4)-(7) are reduced to an equivalent multipoint boundary value problem with parameters

$$\frac{du_r}{dt} = A(t)[u_r(t) + \lambda_r] + f(t), \quad t \in [t_{r-1}, t_r), \ u_r(t_{r-1}) = 0, \quad r = \overline{1,(m+1)l}, \tag{8}$$

$$B_0\lambda_1 + C_0 \lim_{t\to T-0} u_{(m+1)l}(t) + C_0\lambda_{(m+1)l} = d, \tag{9}$$

$$B_i \lim_{t\to t_{il}-0} u_{il}(t) + B_i\lambda_{il} - C_i\lambda_{il+1} = p_i, \quad i = \overline{1,m}. \tag{10}$$

$$\lambda_s + \lim_{t\to t_s-0} u_s(t) = \lambda_{s+1}, \quad s = \{\overline{1,(m+1)l-1}\}\setminus\{il\}, \quad i = \overline{1,m}. \tag{11}$$

If a pair $(\lambda, u[t])$ with

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{(m+1)l})' \in R^{n(m+1)l}, \quad u[t] = (u_1(t), u_2(t), \dots, u_{(m+1)l}(t))'$$

is a solution to problems (8)-(11), then the function system

$$x[t] = (\lambda_1 + u_1(t), \lambda_2 + u_2(t), \dots, \lambda_{(m+1)l} + u_{(m+1)l}(t))'$$

will be a solution to problems (4)-(7).

Vice versa, if $\tilde{x}[t] = (\tilde{x}_1(t), \tilde{x}_2(t), \dots, \tilde{x}_{(m+1)l}(t))'$ is a solution to problems (4)-(7), then $(\tilde{\lambda}, \tilde{u}[t])$ with $\tilde{\lambda} = (\tilde{x}_1(0), \tilde{x}_2(t_1), \dots, \tilde{x}_{(m+1)l}(T))'$,

$\tilde{u}[t] = (\tilde{x}_1(t) - \tilde{x}_1(0), \tilde{x}_2(t) - \tilde{x}_2(t_2), \dots, \tilde{x}_{(m+1)l}(t) - \tilde{x}_{(m+1)l}(t_{(m+1)l-1}))'$ will be a solution to problem (8)-(11). However, problems (8)-(11) differ from (4)-(7) in that there appear the initial conditions at the points $t = t_{r-1}$, $r = \overline{1,(m+1)l}$, which allow us to determine $u_r(t)$, $t \in [t_{r-1}, t_r)$, $r = \overline{1,(m+1)l}$, from the Volterra integral equation of the second kind

$$u_r(t) = \int_{t_{r-1}}^{t} A(\tau)[u_r(\tau) + \lambda_r]d\tau + \int_{t_{r-1}}^{t} f(\tau)d\tau, \quad t \in [t_{r-1}, t_r), \ r = \overline{1,(m+1)l}. \tag{12}$$

Instead of substituting the corresponding right-hand side of (12) and repeating this process ν $(\nu = 1,2,\dots)$ once, we obtain a representation of a function of the form

$$u_r(t) = D_{\nu,r}(t)\lambda_r + F_{\nu,r}(t) + G_{\nu,r}(u,t), \quad t \in [t_{r-1}, t_r), \ r = \overline{1,(m+1)l}, \tag{13}$$

$$D_{,\nu,r}(t)=\sum_{j=0}^{\nu-1}\int_{t_{r-1}}^{t_r}A(\tau_1)\ldots\int_{t_{r-1}}^{\tau_{\nu-2}}A(\tau_{\nu-1})\int_{t_{r-1}}^{\tau_{\nu-1}}A(\tau_\nu)d\tau_\nu\ldots d\tau_1,$$

$$G_{\nu,r}(u,t)=\int_{t_{r-1}}^{t}A(\tau_1)\ldots\int_{t_{r-1}}^{\tau_\nu}A(\tau_{\nu+1})u_r(\tau_{\nu+1})d\tau_{\nu+1}\ldots d\tau_1,$$

$$F_{\nu,rr}(t)=\int_{t_{r-1}}^{t_r}f(\tau_1)d\tau_1+\sum_{t_{r-1}}\int_{t_{r-1}}^{t}A(\tau_1)\ldots\int_{t_{r-1}}^{\tau_{j-2}}A(\tau_{\nu-1})\int_{t_{r-1}}^{\tau_{\nu-1}}f(\tau_\nu)d\tau_\nu\ldots d\tau_1.$$

From (13) we find that

$$\lim_{t\to t_r-0}u_r(t)=D_{\nu,r}(t_r)\lambda_r+F_{\nu,r}(t_r)+G_{\nu,r}(u_r,t_r),\ r=\overline{1,(m+1)l}.$$

Substituting the corresponding right-hand parts (14) in conditions (9),(10), we get the system of equations for unknown parameters $\lambda_1,\lambda_2,\ldots,\lambda_{(m+1)l}$:

$$Q_\nu(l)\lambda=-F_\nu(l)-G_\nu(u,l),\ \lambda\in R^{n(m+1)l}, \tag{15}$$

A solution of the multipoint boundary value problem with parameters (4) - (7) is found as the limit of the sequence of pairs $(\lambda^{(k)},u^{(k)}[t])$, determined by the following algorithm:

Step 0. (a) Assuming that for some $l\in N$, $\nu\in N$, the matrix $Q_\nu(l):R^{n(m+1)l}\to R^{n(m+1)l}$ is invertible, the initial approximation of the parameter $\lambda\in R^{n(m+1)l}$, is found from $Q_\nu(l)\lambda^{(0)}=-F_\nu(l)$, $\lambda^{(0)}=-[Q_\nu(l)]^{-1}F_\nu(l)$;

(b) Using vector components $\lambda^{(0)}\in R^{n(m+1)l}$ and solving the Cauchy problem (8) with $\lambda_r=\lambda_r^{(0)}$ on the subintervals $[t_{r-1},t_r)$, we find functions $u_r^{(0)}(t)$, $r=\overline{1,(m+1)l}$.

Step 1. (a) Substituting found $u_r^{(0)}(t)$ to the right-hand side of (15), from equation $Q_\nu(l)\lambda=-F_\nu(l)-G_\nu(u^{(0)},l)$ we determine the first approximation of the parameter $\lambda^{(1)}$;

(b) Solving the Cauchy problem (8) on the subintervals $[t_{r-1},t_r)$, with $\lambda_r=\lambda_r^{(0)}$, we find functions $u^{(1)}(t)$, $r=\overline{1,(m+1)l}$. And so on.

Continuing the process, on Step k we get the pair system $(\lambda_r^{(k)},u_r^{(k)}(t))$, $k=0,1,2\ldots$, $r=\overline{1,(m+1)l}$.

The following theorem provide sufficient conditions for the feasibility and convergence of the proposed algorithm to a unique solution to the boundary value problem with impulsive effect (1) - (3).

Theorem 1. *Suppose that for some $l\in N$ and $\nu\in N$ the matrix $Q_\nu(l):R^{n(m+1)l}\to R^{n(m+1)l}$ is invertible and the following inequalities hold:*

$$\left\|[Q_\nu(l)]^{-1}\right\|\le\gamma_\nu(l), \tag{16}$$

$$q_\nu(l)=\gamma_\nu(l)\max\left[1,\max_{i=1,m}\frac{h_i}{l}\|B_i\|,\frac{h_{m+1}}{l}\|C_0\|\right]\times \tag{17}$$
$$\times\left\{\exp\left(\frac{\alpha h^0}{l}\right)^j-\sum_{j=0}^{\nu}\frac{1}{j!}\left(\frac{\alpha h^0}{l}\right)^j\right\}<1.$$

Then the boundary value problem with impulse effect (1)-((3) has a unique solution $x^*(t)$ and the following estimation is true:.

$$\|x^*\|=\max_{r=1,(m+1)l}\sup_{t\in[t_{r-1},t_r)}\|x^*(t)\|\le L_{1,\nu}(l)\max(\|d\|,\|f\|_1,\max_{i=1,m}\|p_i\|) \tag{18}$$

where $L_{1,\nu}(l)$ does not depend on $f(t)$, d, p_i and is calculated through B_i, C_i, $i=\overline{(0,m)}$, α, h^0, $\gamma_\nu(l)$, $q_\nu(l)$, ν, l.

The following statements establish that the conditions of Theorem 1 are not only sufficient but necessary for the unique solvability of problems (1) - (3).

Theorem 2. *The boundary value problem with impulse action (1) - (3) is uniquely solvable if and only if for any $l \in N$ there exists $\nu \in N$ such that the matrix $Q_\nu(l): R^{n(m+1)l} \to R^{n(m+1)l}$ is invertible and inequalities (16) and (17) of Theorem 1 are satisfied.*

Theorem 3. *The boundary value problem with impulse action (1) - (3) is uniquely solvable if and only if for any $\nu \in N$ there exists $l = l(\nu) > 0$, $l \in N$, such that the matrix $Q_\nu(l): R^{n(m+1)l} \to R^{n(m+1)l}$ is invertible and inequalities (16) and (17) of Theorem 1 are satisfied.*

The following statements establish the relationship between the constant of the correct solvability and an upper bound of the norm of the matrix $Q_\nu(l)$.

Theorem 4. *If the boundary value problem with impulsive action (1) - (3) is uniquely solvable with a constant K, then for any $\varepsilon > 0, \nu \in N$ there exists $l_1 = l_1(\varepsilon, \nu)$ such that $Q_\nu(l)$ is invertible for any $l \geq l_1(\varepsilon, \nu)$ and the following estimate is valid:*

$$\|[H^{-1}Q_\nu(l)]^{-1}\| \leq (1+\varepsilon)K$$

where

$$H = \frac{1}{l} diag\left(\underbrace{h_{m+1}l, h_1 l, \dots, h_1 l}_{l}, \underbrace{h_1 l, h_2 l, \dots, h_2 l}_{l}, \dots, \underbrace{h_m l, h_{m+1} l, \dots, h_{m+1} l}_{l}\right).$$

Theorem 5. *Let for some $\nu \in N$ there exists $l_0 = l_0(\nu)$ such that for all $l \geq l_0(\nu)$ the matrix $Q_\nu(l)$ is invertible and its inverse satisfies the inequality*

$$\|[H^{-1}Q_\nu]^{-1}\| \leq \gamma,$$

where γ is a constant independent of l. Then problems (1) - (3) are correctly solvable with the constant $K = \gamma$.

One of the main conditions for the feasibility of the algorithm and the unique solvability of the two-point boundary value problem with an impulse effect is the invertibility of the matrix $Q_\nu(l)$ for some ν, l. The block band structure of the matrix $Q_\nu(l)$ allows us to get the recurrence formulas that block-wise determine the elements of $[Q_\nu(l)]^{-1}$. Assuming that the matrices $C_i, i = \overline{(1,m)}$, are invertible, we establish the equivalence of the invertibility of $Q_\nu(l)$ to that of the $(n(m+1)l \times n(m+1)l)$ matrix $M_\nu(l)$ defined as

$$M_\nu(l) = B_0 + C_0 \prod_{s=(m+1)l}^{ml+1} [I + D_{\nu,s}(t_s)] \prod_{j=m}^{1} \left\{ C_j^{-1} B_j \prod_{s=jl}^{(j-1)l+1} [I + D_{\nu,s}(t_s)] \right\}$$

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Тлеулесова А.Б.
Импульстік әсері бар сызықты екінүктелі шеттік есептің бірмәндік шешілімдігі туралы

Аңдатпа: Импульстік әсері бар жай дифференциалдық теңдеулер жүйесі үшін шеттік есеп қарастырылады. Ұсынылып отырған есептің бірмәндік шешілімдігінің қажетті және жеткілікті шарттары тағайындалған.

Түйінді сөздер: шеттік есеп, импульс, параметрлеу тәсілі, бірмәнді шешілімділік

Тлеулесова А.Б.
О корректной разрешимости линейной двухточечной краевой задачи с импульсным воздействием

Аннотация. Рассматривается краевая задача для системы обыкновенных дифференциальных уравнений с импульсным воздействием. Установлены необходимые и достаточные условия корректной разрешимости рассматриваемой задачи.

Ключевые слова: краевая задача, импульс, метод параметризации, корректная разрешимость

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ON ONE APPROACH TO SOLVE A NONLOCAL PROBLEM WITH PARAMETER FOR A SECOND ORDER PARTIAL INTEGRO-DIFFERENTIAL EQUATION OF HYPERBOLIC TYPE

Abstract. *A linear nonlocal problem with a parameter for partial integro-differential equations of hyperbolic type is considered. This problem is investigated by the Dzhumabaev parameterization method. We offer an algorithm for solving nonlocal problems with parameter for partial integro-differential equations of hyperbolic type. First, the original problem is reduce to an equivalent problem consisting a family of boundary value problems for ordinary integro-differential equations with parameters and integral relations. Then, we reduced the family of boundary value problems for ordinary integro-differential equations with parameters to a family of special Cauchy problems for ordinary integro-differential equations with parameters in subdomains and functional relations. At fixed values of parameters the family of special Cauchy problems for ordinary integro-differential equations in subdomains has a unique solution. A system of linear functional equations with respect to parameters is compiled. We propose an algorithm for finding an approximate solution to the equivalent problem. This algorithm includes the approximate solution of the family of Cauchy problems for ordinary differential equations and solving the linear system of functional equations.*

Key words: *nonlocal problem with parameters, partial integro-differential equations of hyperbolic type, family of boundary value problems with parameter, ordinary integro-differential equations, Dzhumabaev parameterization method, algorithm.*

The problem of constructing effective mathematical models finds its solution in many areas of life sciences and technology. A modern approach in the theory of control and identification of